SPATIAL MODELS OF LEGISLATIVE VOTING

By

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Chapter 2: The Geometry of Legislative Roll Call Voting

Overview

In this chapter I show the geometry of legislative roll call voting. To illustrate the geometry I will use the simplest possible spatial model in which each legislator is represented as a point and each roll call is represented as *two points* – one corresponding to the policy outcome associated with "Yea" and one corresponding to the policy outcome associated with "Nay". I assume that legislators always vote for the closest outcome in the space so that they make no voting errors. Hereafter, I will refer to this as *perfect* voting.

In chapter one I outlined a simple theory of spatial maps that I call the basic space theory of ideology. This theory is a parsimonious tool for understanding the construction and interpretation of spatial maps. These maps are based upon a simple Euclidean geometry of points, lines, and planes. The purpose of this chapter is to show the basics of this geometry.

I will analyze the one-dimensional problem first when there is no voting error. After I develop the geometry of perfect voting I will show how the model can be extended to the analysis of interest group ratings and how it compares to the Rasch model used in educational testing and the "Pick-Any-N" model used in Marketing and

Psychometrics. The second half of the chapter will show the geometry in more than one dimension.

The Geometry In One Dimension

Suppose there are p legislators and q roll calls. I assume that each legislator has an ideal point in the policy space represented by the point $\mathbf{X_i}$ with a symmetric single-peaked utility function centered at the ideal point. For example, Figure 2.1 shows the two most common utility functions used in applied work – the normal and the quadratic. The classic normal "bell curve" has the advantage that the utility is always positive and the tails of the distribution asymptote so that alternatives far away from the ideal point have very small utilities. When error is added to the utility function in order to allow for probabilistic voting – the *random utility* model – the choice of the utility function is important. In particular the two utility functions have different implications about how extremists vote. This is especially true in two or more dimensional voting models. Chapter 3 is devoted to this topic.

If there were no error the legislator would vote for the closest alternative in the policy space on every roll call. This results from the fact that the utility function is *symmetric*. Let the two policy outcomes corresponding to Yea and Nay on the jth roll call be represented by $\mathbf{O_{jy}}$ and $\mathbf{O_{jn}}$ respectively. In most cases it is more convenient to work with the midpoint of the two outcomes:

$$\mathbf{Z_j} = \frac{\mathbf{O_{jy}} + \mathbf{O_{jn}}}{\mathbf{2}}$$

In one dimension $\mathbf{Z_j}$ is known as a *cutpoint* that divides the Yeas from the Nays. With perfect spatial voting all the legislators to the left of $\mathbf{Z_j}$ vote for one outcome and all the legislators to the right of $\mathbf{Z_j}$ vote for the opposite outcome. To simplify matters assume that that every legislator to the left of $\mathbf{Z_j}$ votes "Yea" and every legislator to the right of $\mathbf{Z_j}$ votes "Nay". Fixing the *polarity* of the roll calls so that the left outcome is always Yea does not affect any of the analysis below.

Figure 2.2 shows 6 evenly spaced legislators along a dimension with 5 cutting points -- one between each pair of adjacent legislators. The 5 roll calls are shown below the cutpoints. With the legislators and cutpoints arranged in order from left to right and

Figure 2.2 Perfect Spatial Voting in One Dimension

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                -1---------0--------+1
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Legislators	X_1		X_2		X_3		X_4		X_5		X_6
Cutpoints		Z_1		Z_2		Z_3		Z_4		Z_5	
1	Y		N		N		N		N		N
2	Y		Y		N		N		N		N
3	Y		Y		Y		N		N		N
4	Y		Y		Y		Y		N		N
5	Y		Y		Y		Y		Y		N

with Yea always to the left of the cutpoint, the pattern of Y's and N's form a triangle. This pattern of Yeas and Nays is identical to the responses to a classic Guttman scale (Weisberg, 1968). In this regard, the algorithms described below and in Chapter 3 that recover the legislator ideal points and roll call midpoints are quite similar to a Guttman scaling of *dominance* data.[1]

As a practical matter, we would almost never observe real roll call data with the nice triangle pattern. Instead, we are more likely to see roll call data as it is displayed in Figure 2.3. This is the same data as shown in Figure 2.2 only I have changed the order of the legislators and the roll calls. I have also changed the polarity of roll calls 3 and 5 so that Nay rather than Yea is to the left of the cutpoint. The data is also displayed in the conventional manner with the rows as legislators and the columns as roll calls.

Figure 2.3 Recovering the Legislator Points

	Roll Calls				
Legislators	1	3	5	4	2
Five	N	Y	N	N	N
Six	N	Y	Y	N	N
Four	N	Y	N	Y	N
One	Y	N	N	Y	Y
Two	N	N	N	Y	Y
Three	N	N	N	Y	N

Agreement Scores

1.0					
.8	1.0				
.8	.6	1.0			
.2	.0	.4	1.0		
.4	.2	.6	.8	1.0	
.6	.4	.8	.6	.8	1.0

Squared Distances

.00					
.04	.00				
.04	.16	.00			
.64	1.00	.36	.00		
.36	.64	.16	.04	.00	
.16	.36	.04	.16	.04	.00

Double-Centered Matrix

.09					
.15	.25				
.03	.05	.01			
-.15	-.25	-.05	.25		
-.09	-.15	-.03	.15	.09	
-.03	-.05	-.01	.05	.03	.01

Legislator Points

$X_5 = .3$

$X_6 = .5$

$X_4 = .1$

$X_1 = -.5$

$X_2 = -.3$

$X_3 = -.1$

With perfect one-dimensional voting legislator ideal points and roll call cutpoints can always be found that exactly reproduce the roll call data. A formal proof of this is shown in the Appendix to this chapter. Here I will simply show how to recover the legislator and roll call points in one dimension with a simple 4-step process.

The first step is to compute an *agreement score* matrix for the legislators. The agreement score between two legislators is simply the proportion of times they vote the same way over all the roll calls. This matrix is shown below the roll call data in Figure 2.3. (The agreement score matrix is symmetric so I only show the lower triangle in Figure 2.3.) For example, legislators five and six vote the same way except on roll call 5 where legislator five votes Yea and legislator six votes Nay. Hence their agreement score is 4/5 or .8. Similarly, legislators six and four vote the same way on roll calls 1, 3, and 2, and opposite of each other on roll calls 5 and 4. Therefore their agreement score is 3/5 or .6. This score is in the third row and second column of the matrix.

In general, suppose the legislators are ordered 1, 2, 3, …, p, from left to right. Let k_1 be the number of cutpoints between legislators 1 and 2, k_2 be the number of cutpoints between legislators 2 and 3, and so on, with k_{p-1} being the number of cutpoints between legislators p-1 and p. Defined in this way the k's will add up to q; that is

$$\mathbf{q} = \sum_{i=1}^{p-1} \mathbf{k}_i > \mathbf{0} \qquad (2.1)$$

The agreement score between legislators 1 and 2 is simply $\frac{\mathbf{q - k_1}}{\mathbf{q}}$ because 1 and 2 agree on all roll calls except for those with cutpoints between them. Similarly, the agreement score between legislators 1 and 3 is $\frac{\mathbf{q - k_1 - k_2}}{\mathbf{q}}$ and the agreement score between legislators 2 and 3 is $\frac{\mathbf{q - k_2}}{\mathbf{q}}$. In general, for two legislators $\mathbf{X_a}$ and $\mathbf{X_b}$ where a≠b, the agreement score is:

$$\mathbf{A_{ab}} = \frac{\mathbf{q} - \sum_{i=a}^{b-1} \mathbf{k_i}}{\mathbf{q}} \tag{2.2}$$

For our six-person legislature shown in Figure 2.2, there is only one cutpoint between every adjacent pair of legislators so all the k's are equal to one.

The second step to getting the legislator and roll call points is to convert the agreement score matrix into a matrix of *squared distances*.[2] We do this by simply subtracting the agreement scores from one and squaring them. That is:

$$\mathbf{d_{ab}^2} = (\mathbf{1} - \mathbf{A_{ab}})^2 = \left[\mathbf{1} - \frac{\mathbf{q} - \sum_{i=a}^{b-1} \mathbf{k_i}}{\mathbf{q}}\right]^{\mathbf{2}} = \left[\frac{\sum_{i=a}^{b-1} \mathbf{k_i}}{\mathbf{q}}\right]^{\mathbf{2}} \tag{2.3}$$

The squared distances are shown to the right of the agreement scores in Figure 2.3. In the appendix to this chapter I prove that, with perfect one-dimensional voting, these distances are *exact*. That is, there exists a set of points *on a straight line* that *exactly reproduce* these distances *and the roll calls* that are used to create the distances. The *rank ordering* of those points is the same as the rank ordering of the "true" legislator points. In other words, with perfect voting in one dimension you can only recover a rank ordering! It is *impossible* to recover an interval scale. I will illustrate this with some examples after I finish showing how to get the legislator points.

The third step is to *double-center* the matrix of squared distances. That is, from each element of the matrix of squared distances subtract the row mean, subtract the column mean, add the matrix mean, and divide by –2. This produces a cross product matrix of the legislator coordinates (Young and Householder, 1938; Ross and Cliff, 1964). For example, the squared distance between legislator "a" and legislator "b" is:

$$d_{ab}^2 = \left[\frac{\sum_{i=a}^{b-1} k_i}{q}\right]^2 = (X_a - X_b)^2 \qquad (2.4)$$

The mean of row "a" is

$$\frac{\sum_{i=1}^{p} d_{ai}^2}{p} = X_a^2 - 2X_a\bar{X} + \frac{\sum_{i=1}^{p} X_i^2}{p}$$

where $\bar{\mathbf{X}}$ is the mean of the p **X**'s. The mean of column "b" is

$$\frac{\sum_{i=1}^{p} d_{ib}^2}{p} = X_b^2 - 2X_b\bar{X} + \frac{\sum_{i=1}^{p} X_i^2}{p}$$

and the mean of the matrix is

$$\frac{\sum_{i=1}^{p}\sum_{\ell=1}^{p} d_{i\ell}^2}{p^2} = \frac{\sum_{\ell=1}^{p} X_\ell^2}{p} - 2\bar{X}^2 + \frac{\sum_{i=1}^{p} X_i^2}{p}$$

Subtracting the row and column means and adding the matrix mean cancels all the squared terms leaving only the cross products. To see this:

$$\left\{(X_a^2 - 2X_aX_b + X_b^2) - (X_a^2 - 2X_a\bar{X} - \frac{\sum_{i=1}^{p} X_i^2}{p}) - (X_b^2 - 2X_b\bar{X} - \frac{\sum_{i=1}^{p} X_i^2}{p}) + \right.$$

$$\left.\left(\frac{\sum_{\ell=1}^{p} X_\ell^2}{p} - 2\bar{X}^2 + \frac{\sum_{i=1}^{p} X_i^2}{p}\right)\right\} \div (-2) = (X_a - \bar{X})(X_b - \bar{X})$$

If we adopt the restriction that legislator coordinates must add up to zero, that is $\bar{\mathbf{X}} = \mathbf{0}$, then the entries of the double centered matrix are simply $\mathbf{X_a X_b}$ for all pairs of legislators. The diagonal terms are simply the legislator coordinates squared; that is, $\mathbf{X_a^2}$.

The final step is to take the square root of a diagonal element of the double centered matrix and then divide through the corresponding column of the matrix by this square root. Using the first diagonal element produces the legislator coordinates as shown in Figure 2.3. In sum, to get the legislator coordinates:

1. Compute the p by p agreement score matrix
2. Convert the agreement score matrix into a matrix of squared distances
3. Double Center the matrix of squared distances
4. Take the square root of a diagonal element of the double centered matrix and divide it through the corresponding column

Figure 2.3 shows that an interval level set of points is recovered. However, *this is an artifact of the distribution of cutting points*. For example, if $k_1 > k_2$, this has the effect of making $d_{12} > d_{23}$ even if the true coordinates $\mathbf{X_1}$, $\mathbf{X_2}$, $\mathbf{X_3}$ were *evenly spaced*. For example, Figure 2.4 repeats the steps shown in Figure 2.3 with exactly the same set of roll calls only now instead of one roll call cutpoint between $\mathbf{X_1}$ and $\mathbf{X_2}$ there are four cutpoints between them; that is, $k_1 = 4$ so that q=8. In other words, the first roll call is repeated four times. The agreement score matrix is symmetric so I only show the lower triangle in Figure 2.4.

With one cutpoint between each pair of legislators the recovered legislator coordinates were evenly spaced: -.5, -.3 -.1, +.1, +.3, +.5. Placing three additional cutpoints between $\mathbf{X_1}$ and $\mathbf{X_2}$ pushes them apart in the recovered coordinates:

Figure 2.4 The Effect of the Number of Cutpoints

Agreement Scores

1.000					
.875	1.000				
.875	.750	1.000			
.125	.000	.250	1.000		
.625	.500	.750	.500	1.000	
.750	.625	.875	.375	.875	1.000

Squared Distances

0.000000					
0.015625	0.000000				
0.015625	0.062525	0.000000			
0.765625	1.000000	0.562500	0.000000		
0.140625	0.250000	0.062500	0.250000	0.000000	
0.062525	0.140625	0.015625	0.390625	0.015625	0.000000

Double-Centered Matrix

0.062500					
0.093750	0.140625				
0.031250	0.046875	0.015625			
-0.156250	-0.234375	-0.078125	0.390625		
-0.031250	-0.046875	-0.015625	0.078125	0.015625	
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Legislator Points

$X_5 = .250$

$X_6 = .375$

$X_4 = .125$

$X_1 = -.625$

$X_2 = -.125$

$X_3 = .000$

-.625, -.125, .000, +.125, +.250, +.375. However, both sets of legislator points produce the same set roll call votes. With perfect voting the two configurations produce the same voting patterns.

In sum, with perfect one dimensional voting, the legislator configuration is only identified up to a *weak monotone transformation of the true rank ordering*. "Weak monotone" means that if there are no cutpoints between a pair of legislators – some $k_i = 0$ – then those legislators will be recovered in the same position – a tied rank -- because their voting patterns will be identical. However, if there are cutting points between every pair of adjacent legislators, that is, $k_i \geq 1$ for i=1, …, p-1, then *the true rank ordering is recovered*. Also note that the *mirror image* of the recovered rank ordering is also a solution. Taking the mirror image of the rank ordering and reversing the polarity assigned to each cutpoint produces the same roll call votes.

Interest Group Ratings

When an interest group like the Americans for Democratic Action (ADA) issues ratings of members of Congress or members of a state legislature[3] they typically select about ten to twenty roll calls that they view as important and they compute a simple agreement score between how they would have voted with how the members actually voted. The ratings are typically reported in the form of percentages so that they range from 100 indicating a perfect rating to 0 meaning that the member is under the influence of the forces of darkness. For example, if the ADA rating is used as a measure of liberalism, then that means that a legislator receiving a rating of 100 is more liberal than a legislator receiving a rating of 90 who in turn is more liberal than someone receiving a rating of 80, and so on. Used in this way a researcher is implicitly assuming that the interest group is *exterior to the legislators*. That is, the assumption is that a member can't be *more liberal than the ADA* (no one can get a rating of 110!).

In the context of the one-dimensional spatial model developed above, this means that the interest group is at the end of the dimension *exterior to the legislators and the cutpoints of the roll calls it chooses for its ratings*. An example of this is shown in Figure 2.5.

Figure 2.5 Interest Group Ratings in One Dimension

```
            Liberal                      Conservative
            -1----------------0----------------+1
Legislators ADA X1  X2  X3  X4  X5  X6  X7 ... Xp-1 Xp
Cutpoints               Z1      Z2  Z3  Z4 ...Z20
```

Suppose Yea is always the outcome to the left of a midpoint and Nay is the outcome to the right and there is perfect spatial voting. On the twenty roll calls selected by the ADA it and legislators $\mathbf{X_1}$ and $\mathbf{X_2}$ would vote Yea so that legislators $\mathbf{X_1}$ and $\mathbf{X_2}$ would receive ratings of 100. Legislators $\mathbf{X_3}$ and $\mathbf{X_4}$ vote Nay on the first roll call and Yea on the other nineteen so they would receive a 95 rating. As the number of cutpoints between a legislator and the ADA increases, the rating decreases. Because the ADA only chooses twenty roll calls to compute its ratings, the ratings will be very coarse with many legislators receiving the same rating. This is likely to be quite severe at the ends of the dimension with many legislators receiving 100's and 0's. In addition, not only will the ratings be quite coarse, the example at the end of the previous section shows that the ratings will be very sensitive to the *distribution* of the cutpoints (Snyder, 1992; Poole and Rosenthal, 1997). For example, if the cutpoints are concentrated near the middle of the distribution of legislators then there will be an abundance of 100 and 0 ratings.

The assumption that legislators cannot be more liberal than the ADA leads to the simple spatial model shown in Figure 2.5 where the interest group is at the end of the dimension exterior to the legislators and the cutpoints of the roll calls it chooses for its ratings. However, this is an *assumption* that should be tested. For example, Figure 2.6 shows a configuration of legislators, cutpoints and the ADA that produces exactly the same ratings as the configuration shown in Figure 2.5. In Figure 2.6 the Yea outcome on the first roll call is to the right of the cutpoint while the Yea outcome is to the left of the other nineteen cutpoints as in Figure 2.5. With perfect spatial voting, the ADA gives ratings of 100 to legislators $\mathbf{X_1}$ and $\mathbf{X_2}$. Legislator $\mathbf{X_3}$ votes Nay on the first roll call and Yea on the other nineteen so she receives a 95 rating. Legislator $\mathbf{X_4}$ votes Yea on the first roll call, Nay on the second roll call, and Yea on the other eighteen so she would also receive a 95 rating.

Figure 2.6 Unfolded Interest Group Ratings

Liberal **Conservative**

-1-----------------0----------------+1

Legislators X_3 X_1 **ADA** X_2 X_4 X_5 X_6 X_7 ... X_{p-1} X_p

Cutpoints Z_1 Z_2 Z_3 Z_4 Z_5 ... Z_{20}

The spatial configurations in Figures 2.5 and 2.6 produce exactly the same ADA ratings. If Figure 2.6 is the true configuration, then the ADA ratings are not an accurate measure of liberalism. Legislator $\mathbf{X_3}$ is the most liberal, but she receives a rating lower than legislators $\mathbf{X_1}$ and $\mathbf{X_2}$ who are adjacent to the ADA. Adding to the distortion, legislator $\mathbf{X_4}$ who is towards the interior of the dimension receives the same rating as the most liberal legislator, $\mathbf{X_3}$.

If Figure 2.6 is the true configuration, then Figure 2.5 is an example of a *folded dimension* (Coombs, 1964, ch. 5). That is, if the dimension in Figure 2.6 is a string that we can pick up at the position of the ADA, then the left end *folds back* onto the dimension and we get the configuration in Figure 2.5. In order to get the true dimension the ADA ratings have to be *unfolded*.[4]

To reiterate, when interest group ratings are used as measures of some dimension like liberal-conservative or environmentalism, the implicit assumption is that the interest group is at the end of the dimension exterior to the legislators and the cutpoints of the roll calls it chooses for its ratings. This assumption can be tested by treating the interest group as if it were a member of Congress (Poole and Rosenthal, 1997, ch. 8).

The Rasch Model From Educational Testing

The simple configuration of legislator ideal points and roll call cutpoints shown in Figure 2.2 can also be interpreted in an educational testing context. Suppose the dimension corresponds to some latent ability – for example, the ability to solve logical puzzles. Then $\mathbf{X_i}$ is the individual test taker's level of ability and $\mathbf{Z_j}$ is the level of difficulty of the test question. Suppose the dimension ranges from low ability (-1) to high ability (+1). In this framework a "Y" corresponds to "wrong answer" and an "N" corresponds to "correct answer". $\mathbf{X_1}$ is unable to solve any of the problems while $\mathbf{X_6}$ solves all five of the problems.

The testing model developed by Rasch (1961) is mathematically equivalent to the basic spatial model shown in Figure 2.2 if legislators have quadratic utility functions with additive random error. Since this will be an important topic in Chapter 3, I will show the relationship here for the simple one-dimensional case with no error. In Chapter 3, I will discuss the multiple dimension case with error.

In the quadratic utility model, the utility of the ith legislator for the Yea and Nay alternatives is simply the negative of the squared distance from the legislator's ideal point to the alternatives:

$$\mathbf{U_{iy} = -(X_i - O_{jy})^2} \text{ and } \mathbf{U_{in} = -(X_i - O_{jn})^2}$$

In the spatial voting model, if $\mathbf{U_{iy}} > \mathbf{U_{in}}$ the legislator votes Yea. Stated another way, if the difference, $\mathbf{U_{iy} - U_{in}}$, is positive, the legislator votes Yea. Algebraically:

$$\mathbf{U_{iy} - U_{in} = -(X_i - O_{jy})^2 + (X_i - O_{jn})^2 = 2X_iO_{jy} - O_{jy}^2 - 2X_iO_{jn} + O_{jn}^2 =}$$

$$\mathbf{2X_i(O_{jy} - O_{jn}) - (O_{jy}^2 - O_{jn}^2) = 2\gamma_j(X_i - Z_j)} \qquad (2.5)$$

where $\mathbf{\gamma_j = (O_{jy} - O_{jn})}$ and $\mathbf{2Z_j = (O_{jy} + O_{jn})}$. With perfect voting the legislator and the chosen outcome are on the same side of the midpoint. Hence:

$$\begin{array}{l} \text{if } \mathbf{2\gamma_j (X_i - Z_j) > 0} \text{ Vote Yea} \\ \text{if } \mathbf{2\gamma_j (X_i - Z_j) < 0} \text{ Vote Nay} \end{array} \qquad (2.6)$$

If $O_{jy} > O_{jn}$ this decision rule simplifies to:

$$\text{if } \mathbf{X_i - Z_j > 0} \text{ Vote Yea}$$

$$\text{if } \mathbf{X_i - Z_j < 0} \text{ Vote Nay}$$

The corresponding formulation for the Rasch model is:

$$\begin{aligned} &\text{if } \alpha_j \mathbf{X_i} - \beta_j > \mathbf{0} \text{ correct answer} \\ &\text{if } \alpha_j \mathbf{X_i} - \beta_j < \mathbf{0} \text{ wrong answer} \end{aligned} \qquad (2.7)$$

where α_j is the *item discrimination parameter* and β_j is the *difficulty parameter*. If the test question is clearly stated so that there is no ambiguity then by convention $\alpha_j = 1$. A poorly constructed and ambiguous question would have an α_j near 0. β_j is simply the level of difficulty on the latent dimension.

Although the two models look very much alike – especially with the $\alpha_j = 1$ – they are analytically *very different*. The spatial voting model is a model of *choice* between two alternatives while the testing model is concerned with *ability*. These differences will be more apparent when error is added to the models in Chapter 3.

The Pick-Any-N Data Model From Marketing

Pick-Any-N choice data is used in the marketing field to analyze consumer buy/not buy choices for a range of products. For example, a set of consumers are asked if they drink a number of soft drinks – “do you drink Pepsi?”, “do you drink Coke?”, “do you drink Royal Crown Cola?”, etc. The consumers simply answer “Yes” or “No”. The dimension corresponds to some attribute of the products – for example, level of sweetness. Analytically, in one dimension, this data is equivalent to roll call data if the products are treated as the legislators – the $\mathbf{X_i}$ – and the consumers are treated as the roll call midpoints – the $\mathbf{Z_j}$. The consumer approves, would buy, or use all the products below her location and does not approve, would not buy, or not use all the products above her location.

In more than one dimension the consumer is represented as a *vector* and the products are represented by points. This vector is perpendicular to a cutting plane that separates the products the consumer approves from those she does not approve. This geometry is the same as that for the roll call voting problem and I discuss it in detail below.

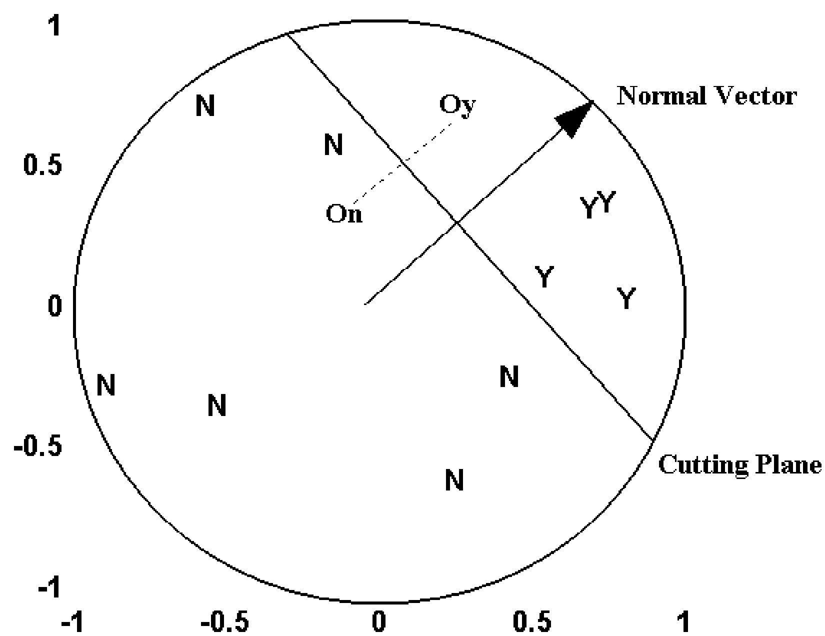
Summary: One Dimensional Perfect Voting

With perfect one-dimensional roll call voting legislator ideal points and roll call cutpoints can always be found that exactly reproduce the roll call data. However, only a *rank-ordering* of the legislators can be recovered. Recovering interval level information from the perfect one-dimensional roll call data is impossible. In addition, with perfect voting/choice/responses in one dimension, the roll call voting problem, the educational testing problem, and the Pick-Any-N data problem are *observationally equivalent*.

The Geometry In More Than One Dimension

In one-dimensional perfect roll call voting both the legislators and the roll call midpoints are represented by points – the $\mathbf{X_i}$ and the $\mathbf{Z_j}$ -- and a joint rank ordering of the legislators and roll call midpoints can be found that exactly reproduces the roll call votes. In two or more dimensional perfect voting a legislator is still represented by a point -- $\mathbf{X_i}$ – but a roll call is now represented by a *plane* that is perpendicular to a line joining the Yea and Nay policy points -- $\mathbf{O_{jy}}$ and $\mathbf{O_{jn}}$ -- and passes through the midpoint, $\mathbf{Z_j}$. The *normal vector* to this *cutting plane* is parallel to the line joining the Yea and Nay policy points. Figure 2.7 shows a simple example of ten legislators in two dimensions.

Figure 2.7. Ten Point Example
Original Positions in Two Dimensions



In Figure 2.7 the legislators are displayed as "N"s or "Y"s corresponding to their votes on the hypothetical roll call. The cutting plane divides the legislators who vote Yea from those who vote Nay and the line joining the Yea and Nay outcomes is parallel to the normal vector. Note that, in the case of perfect voting, the policy points are not identified – any pair of points on a line perpendicular to the plane that are on opposite sides and equidistant from the plane would produce the same pattern of votes. However, the cutting plane is identified up to a region of the space that divides the Yeas from the Nays. For example, the cutting line in Figure 2.7 could be rotated until it was perpendicular to the first dimension.

Technically, even through I treated legislators as *specific* points in the one dimensional perfect voting problem, each legislator could be anywhere in the region between the corresponding pair of roll call midpoints. For example, in Figure 2.2 $\mathbf{X_2}$ could be located anywhere between $\mathbf{Z_1}$ and $\mathbf{Z_2}$. The counterpart to this in more than one dimension is that the legislator can be anywhere in an area/volume bounded by a set of cutting lines/planes. These regions are known as *polytopes*. A two dimensional polytope is a polygon -- an area of a two dimensional space that is bounded by line segments. A three dimensional polytope is a polyhedron -- a volume of three dimensional space that is bounded by two dimensional polygons.

For example, in two dimensions, if a variety of voting coalitions form amongst the legislators, then the q cutting lines will criss-cross one another in a myriad of directions creating a very large number of polytopes in the plane (see Figure 2.8 below). Indeed, Coombs (1964, p. 262) shows that q roll calls create a maximum of $\sum_{\mathbf{k=0}}^{\mathbf{s}} \binom{\mathbf{q}}{\mathbf{k}}$ polytopes where **s** is the number of dimensions. For two dimensions, 1 + q + q(q-1)/2 polytopes are possible with each polytope corresponding to a voting pattern on the q roll calls – e.g., yynynyy**…**etc. If all the possible polytopes are present I will refer to the corresponding configuration of cutting lines/planes as a *Coombs mesh*.

Figure 2.8 shows two simple Coombs meshes for five roll calls (q=5) in two dimensions (s=2). The five cutting lines are numbered at the rim of the circles. The "Y" and "N" on either side of each cutting line indicate how a legislator on that side of the cutting line should vote. The maximum number of polytopes created by five cutting lines in two dimensions is 16 and each of these 16 polytopes corresponds to a unique vector of votes. To reduce clutter, Figure 2.8 only shows the vectors of votes for 6 polytopes in each mesh. Note that the polytopes vary considerably in size from the small triangles and parallelograms in the center of the meshes to the larger "polytopes" at the edges – for example, nynnn in both meshes. (Technically, the region containing nynnn in both meshes is not a polytope because its outer boundary is an arc. The circles are just a graphical convenience to indicate that beyond it there are no intersection points.)

Figure 2.8A 1st Coombs Mesh

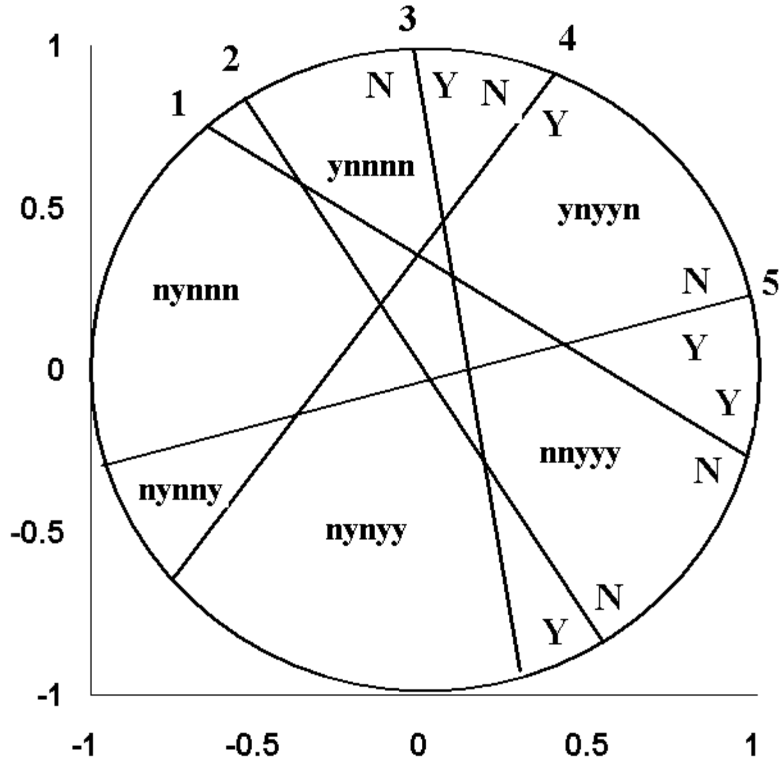
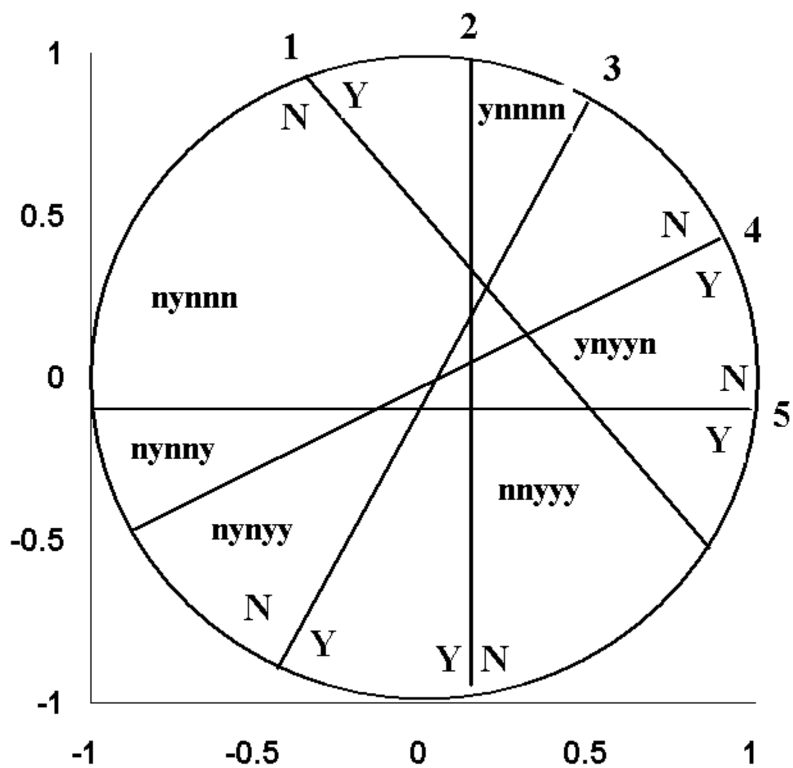


Figure 2.8B 2nd Coombs Mesh



In order to get the maximum number of polytopes for q roll calls in two dimensions every cutting line must intersect every other cutting line. There will be [q(q-1)]/2 intersection points and they must be unique for there to be the maximum of 1 + q + q(q-1)/2 polytopes. In addition, [(q-2)(q-1)]/2 of the polytopes will be in the interior of the Coombs mesh and 2q polytopes will be distributed around the edge of the mesh – that is, they will be *open*.[5] For example, the region containing nynnn in both meshes is open in that beyond the arbitrary circle boundary, there are no more intersection points. The region is infinite. In terms of the coordinate system shown in Figure 2.8, any point with second dimension coordinate of 0.0 and first dimension coordinate between -.5 and -∞ corresponds to nynnn in both meshes.

With q roll calls there are q! ways to number the roll calls and 2^q polarity possibilities for a total of $q!2^q$ combinations of polarity and numbering. Given a numbering and polarity of the roll calls and q(q-1)/2 unique intersection points, there are only 1 + q + q(q-1)/2 unique patterns of votes corresponding to the 1 + q + q(q-1)/2 polytopes. For example, the two meshes in Figure 2.8 have the same numbering of the 5 roll call cutting lines clockwise around the mesh and the same polarity on each roll call. Consequently, the 10 open polytopes around the edge of the mesh have exactly the same patterns. However, the 6 interior polytopes differ in their patterns because the 10 interior intersection points are arranged differently.

This simple example illustrates the complexity of the perfect voting problem. Even if the numbering and the polarity of the roll calls is fixed, there are multiple possible arrangements of the (q-2)(q-1)/2 interior polytopes and each arrangement produces a different roll call matrix. For example, Figure 2.9 shows the roll call matrices for the two meshes in Figure 2.8. The first 10 rows correspond to the 10 open polytopes clockwise around the meshes and the last 6 rows correspond to the 6 interior polytopes. Note that 3 of the 6 interior polytopes of the two meshes have the same roll call voting patterns (rows 11, 13, and 15). Hence, if there were only 13 legislators with perfect voting on five roll calls in two dimensions and our matrix consisted of rows 1 to 11, 13, and 15, either mesh in Figure 2.8 would give us a perfect solution.

Figure 2.9 Roll Call Matrices From Figure 2.8

	Roll Calls									
	Fig. 2.8A					Fig. 2.8B				
Legislators	1	2	3	4	5	1	2	3	4	5
1	Y	Y	N	N	N	Y	Y	N	N	N
2	Y	N	N	N	N	Y	N	N	N	N
3	Y	N	Y	N	N	Y	N	Y	N	N
4	Y	N	Y	Y	N	Y	N	Y	Y	N
5	Y	N	Y	Y	Y	Y	N	Y	Y	Y
6	N	N	Y	Y	Y	N	N	Y	Y	Y
7	N	Y	Y	Y	Y	N	Y	Y	Y	Y
8	N	Y	N	Y	Y	N	Y	N	Y	Y
9	N	Y	N	N	Y	N	Y	N	N	Y
10	N	Y	N	N	N	N	Y	N	N	N
11	N	N	N	N	N	N	N	N	N	N
12	Y	N	N	Y	N	N	N	Y	N	N
13	N	N	Y	Y	N	N	N	Y	Y	N
14	N	N	N	Y	Y	N	Y	Y	Y	N
15	N	Y	N	Y	N	N	Y	N	Y	N
16	N	N	N	Y	N	N	Y	Y	N	N

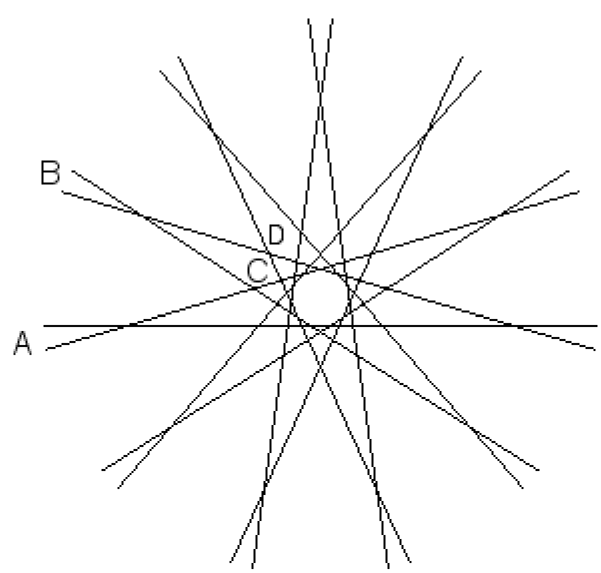
In practice, the number of legislators will be small compared to the number of polytopes so that there will be many "empty" polytopes. For example, in recent U.S. Senates there have been at least 500 non-unanimous roll calls. With 500 roll calls in two dimensions there are 125,251 polytopes and only 100 Senators. This disparity is *typical* of real world sized roll call matrices. Most of the polytopes in any estimated mesh will be empty.

Empty polytopes are also required to reproduce the actual Yea versus Nay margins on roll calls. For example, every roll call in the first mesh in Figure 2.8 has six polytopes on one side of the cutting line and ten polytopes on the opposite of the cutting line. Hence, if there was a legislator assigned to each polytope all five roll calls would divide 10 - 6. In contrast, in the second mesh there are five polytopes on one side of the cutting lines for roll calls one and five and eleven on the opposite side. If there was a legislator assigned to each polytope for the second mesh the margins of the roll calls – ignoring which side is Yea/Nay -- would be 11 - 5, 8 - 8, 9 - 7, 8 – 8, and 11 – 5, respectively.

This simple example makes it clear that the *minimum* number of polytopes on one side of a cutting line is q. For 11 roll calls there are a maximum of 67 polytopes so that if a legislator was assigned to each polytope then the most lopsided a roll call could be is 56 to 11. In real world legislative voting very lopsided votes, for example, 99% to 1%, frequently occur. Consequently, for these to be present in perfect voting the number of legislators must be much smaller than the number of polytopes so that there are enough empty polytopes on one side of a roll call cutting line to produce a lopsided vote. For example, in Figure 2.8B if there was only one legislator on the Yea side of the cutting line for the first roll call and all 11 polytopes on the Nay side of the cutting line contained a single legislator each, then the margin would be 11 Nays and 1 Yea.

Adding to the difficulty of the two or more dimensional perfect voting problem is the fact that the number of cutting lines/planes between pairs of legislators cannot be treated as Euclidean distances. In one-dimensional perfect voting the number of cutpoints between a pair of legislators could be treated as the *Euclidean distance* between two points representing the legislators. Furthermore, if there were three cutpoints between legislators A and B and three cutpoints between legislators C and D, then the points representing A and B would be three units apart and the points representing C and D would also be three units apart. This is not true of two or more dimensional perfect voting. For example, Figure 2.10 shows a *symmetric* Coombs mesh for 11 roll calls in two dimensions. (The mesh is symmetric in that the normal vectors for the cutting lines are evenly spaced from $\pi/2$ to $-\pi/2$ radians.) The pair of legislators in the open polytopes labeled A and B have two cutting lines between them and the pair of legislators in the polytopes labeled C and D also have two cutting lines between them. In addition, C is two cutting lines away from A and B. Because the number of cutting lines between pairs of legislators cannot be treated as Euclidean distances, the method used to solve the one dimensional perfect voting problem (analyzing the double-centered matrix of squared distances) cannot be used to solve the higher dimensional problem.

Figure 2.10 Symmetric Coombs Mesh



Summary: Perfect Voting in More than One Dimension

In summary, with perfect voting in two or more dimensions a solution can always be found by constructing and searching through Coombs meshes. This would be a computational nightmare, however. For every $q!2^q$ possible combinations of roll call numbering and polarity, there are a myriad of Coombs meshes like those shown in Figure 2.8 for two dimensions. Unfortunately, a brute force searching approach is simply not practical. The Coombs mesh for 500 roll calls in two dimensions would have 125,251 polytopes and this number explodes to 20,833,751 in three dimensions. The number of combinations of roll call numbering and polarity is $500!2^{500}$ which is an unimaginably large number on the order of 10^{1000} .

Clearly, the exhaustive search approach is not practical with current computers. It may well be in the future.

The Relationship to the Geometry of Probit and Logit

The complexity of the roll call voting problem in more than one dimension would be greatly simplified if we knew *a priori* the legislator ideal points. The problem would then simplify to finding the cutting planes vis a vis the legislator ideal points. In psychometrics this is known as an *external analysis* (Carroll, 1972) or an *exernal unfolding* (Borg and Groenen, 1997). For example, suppose we are given the configuration of ideal points shown in Figure 2.7 and the corresponding Yeas and Nays for that particular roll call. In an external analysis we take the $\mathbf{X_i}$'s as given and we estimate the cutting plane for the roll call. If error is present then the problem of estimating the cutting plane is equivalent to a probit or logit analysis depending upon the assumptions made about the error. To simplify the presentation below, I will continue to assume that voting is perfect even though both probit and logit "blow up" when there is no error. My aim is to show the geometry of cutting planes between choices. The presence of error only affects the *placement* of the cutting planes. It does not affect the fundamental geometry of the planes themselves. I will discuss the former in chapter 3. Here I only discuss the geometry itself.

Figure 2.11. Ten Point Example
Normal Vector and Projection Line

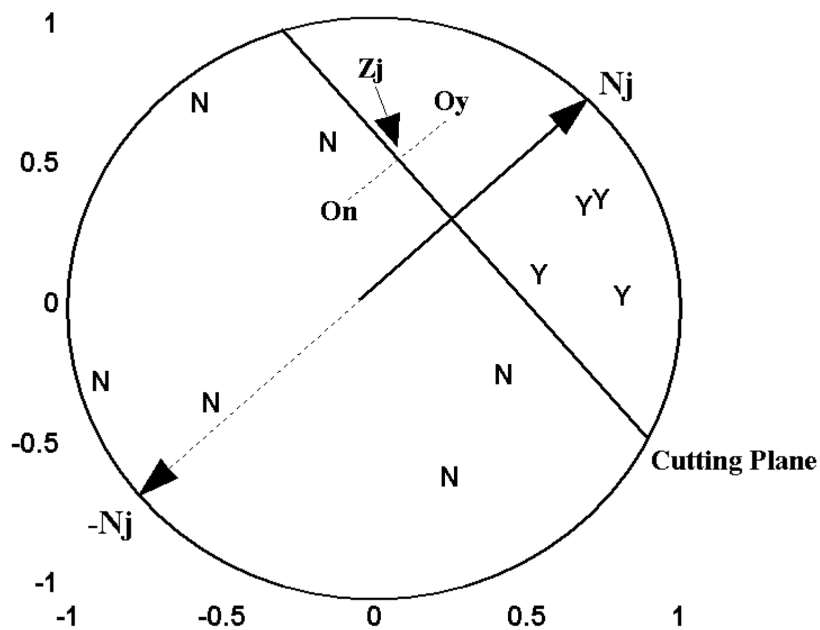


Figure 2.11 shows the same configuration of legislators as Figure 2.7 and illustrates the geometry of a roll call vote in two dimensions. The normal vector is denoted as $\mathbf{N_j}$ and its *reflection* as $\mathbf{-N_j}$. The normal vector is perpendicular to the cutting plane. The cutting plane in two dimensions is defined by the equation

$$\mathbf{N_{j1}(Y_1 - Z_{j1}) + N_{j2}(Y_2 - Z_{j2}) = 0}$$

Where $\mathbf{N_{j1}}$ and $\mathbf{N_{j2}}$ are the components of the normal vector and $\mathbf{(Z_{j1}, Z_{j2})}$ is the midpoint of the roll call outcomes (see Figure 2.11). Any point, $(\mathbf{Y_1}, \mathbf{Y_2})$, that satisfies the above equation lies on the cutting plane. For example, if the normal vector is (3, -2) and the roll call midpoint is (1, 0), then this produces the equation:

$3(y_1 - 1) - 2(y_2 - 0) = 3y_1 - 3 - 2y_2 = 3y_1 - 2y_2 - 3 = 0$ or

$3y_1 - 2y_2 = 3$

so that (0, -3/2), (1/3, -1), (2, 3/2), etc., all lie on the plane.

In three dimensions the cutting plane is defined by the equation

$$\mathbf{N_{j1}(Y_1 - Z_{j1}) + N_{j2}(Y_2 - Z_{j2}) + N_{j3}(Y_3 - Z_{j3})} = 0$$

Where, as above, any point, $(\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y_3})$, that satisfies the above equation lies on the cutting plane. For example, if the normal vector is (2, -2, 1) and the roll call midpoint is (1, 1, 1), then this produces the equation:

$2(y_1 - 1) - 2(y_2 - 1) + (y_3 - 1) = 2y_1 - 2y_2 + y_3 - 1 = 0$ or

$2y_1 - 2y_2 + y_3 = 1$

so that (0, 0, 1), (1, 0, -1), (0, 1, 3), etc., all lie on the plane.

For s dimensions the cutting plane is defined by the vector equation

$$\mathbf{N_j'(Y - Z_j) = 0} \qquad (2.8)$$

Where $\mathbf{N_j}$, $\mathbf{Y}$, and $\mathbf{Z_j}$ are vectors of length s and $\mathbf{0}$ is a s length vector of zeroes.

In general, if $\mathbf{Y_A}$ and $\mathbf{Y_B}$ are both points in the plane then, $\mathbf{Y_A'N_j} = \mathbf{Y_B'N_j} = \alpha_\mathbf{j}$, where $\alpha_\mathbf{j}$ is a scalar constant. Geometrically, every point in the plane projects onto the same point on the line defined by the normal vector, $\mathbf{N_j}$ and its reflection $-\mathbf{N_j}$ (see Figure 2.11). This projection point is

$$\mathbf{M_j} = \alpha_j \frac{\mathbf{N_j}}{\sum_{k=1}^{s} \mathbf{N}_{jk}^2} \qquad (2.9)$$

Note that, by construction, $\mathbf{M_j}'\mathbf{N_j} = \alpha_j$. In addition, because the midpoint of the Yea and Nay policy points, $\mathbf{Z_j}$, is on the cutting plane, it also projects to the point $\mathbf{M_j}$. The cutting plane passes through the line formed by the normal vector and its reflection (see Figure 2.11) at the point $\mathbf{M_j}$.

In the case of a simple probit analysis the cutting plane consists of all possible legislator ideal points such that the probability of the corresponding legislator voting Yea or voting Nay is exactly .5; namely:

P(legislator i votes Yea) = P(legislator i votes Nay) =

$$\Phi\left(\frac{\beta_0 + \beta_1\mathbf{X}_{i1} + \beta_2\mathbf{X}_{i2} + \ldots + \beta_s\mathbf{X}_{is}}{\sigma}\right) = 1 - \Phi\left(\frac{\beta_0 + \beta_1\mathbf{X}_{i1} + \beta_2\mathbf{X}_{i2} + \ldots + \beta_s\mathbf{X}_{is}}{\sigma}\right) =$$

$$\Phi(\mathbf{0}) = .5$$

Where $\Phi(.)$ is the distribution function for the normal and $\mathbf{X}_{i1}$, $\mathbf{X}_{i2}$, …, $\mathbf{X}_{is}$ are legislator i's coordinates on the s dimensions. Because the β's and σ cannot be separately identified, the usual assumption is to set $\sigma = 1$. The above equation reduces to:

$$\beta_0 + \beta_1\mathbf{X}_{i1} + \beta_2\mathbf{X}_{i2} + \ldots + \beta_s\mathbf{X}_{is} = \mathbf{0} \quad \text{or}$$

$$\beta_1\mathbf{X}_{i1} + \beta_2\mathbf{X}_{i2} + \ldots + \beta_s\mathbf{X}_{is} = \mathbf{X}_i'\tilde{\beta} = -\beta_0$$

where $\mathbf{X_i}$ is the s length vector of legislator coordinates and $\tilde{\beta}$ is a s length vector of the coefficients β_1 , β_2 , β_3 , …, β_s . Note that the expression $\mathbf{X}_i'\tilde{\beta} = -\beta_0$ is exactly the same as $\mathbf{Y}'\mathbf{N_j} = \alpha_j$ which was used above. Namely, set $\mathbf{N_j} = \tilde{\beta}$ and every point in the plane projects onto the point:

$$\mathbf{M_j} = -\beta_0 \frac{\tilde{\beta}}{\sum_{k=1}^{s} \tilde{\beta}_k^2}. \qquad (2.10)$$

In other words, in a regular probit context the coefficients on the independent variables form a normal vector to a plane that passes through the point $-\beta_0 \frac{\tilde{\beta}}{\sum_{k=1}^{s} \tilde{\beta}_k^2}$.

The simple logit case is identical to probit. The logit probabilities are:

$$\frac{e^{(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_s X_{is})}}{1 + e^{(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_s X_{is})}} = \frac{1}{1 + e^{(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_s X_{is})}} = .5$$

Canceling out the denominator and taking the natural log of both sides yields the same equation as probit:

$$\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_s X_{is} = 0$$

In both probit and logit the coefficients on the independent variables form a normal vector to a plane that passes through the point $-\beta_0 \frac{\tilde{\beta}}{\sum_{k=1}^{s} \tilde{\beta}_k^2}$. The cosine of the angle between the normal vectors from probit and logit should be very close to one. That is:

$$|\cos\theta| = \left| \frac{\tilde{\beta}_P' \tilde{\beta}_L}{\|\tilde{\beta}_P\| \|\tilde{\beta}_L\|} \right| \approx 1$$

where θ is the angle between the two normal vectors, and ||.|| is the corresponding norm of the normal vector. This is a useful check on the two estimation techniques.

Although I am limiting my analysis in this book to the simple spatial model of only two outcomes per roll call, the geometry of more than two choices is a simple extension of the above. For example, in ordered probit or ordered logit there is only one normal vector but multiple cutting planes. In multinomial probit or multinomial logit, there are multiple normal vectors. For example, if there are three choices there are two normal vectors. The cutting planes that divide the choices are easily derived using simple vector geometry.

Conclusion

Understanding the basic geometry of roll call voting is an essential foundation for realistic models of choice that allow for error. Although there is no simple solution for the perfect voting problem in more than one dimension, in applied work this is not a problem. In Chapter 3 I show a method – Optimal Classification (OC) – that is based upon the geometry shown above that is designed to analyze real-world roll call data. In most circumstances it is not necessary for every roll call cutting line/plane to intersect all the others. A true Coombs mesh is a rarity. When the number of cutting lines/planes is over 100 in two or more dimensions the space is chopped up so finely by the criss-crossing that the legislator polytopes are, for all practical purposes, points (see Figure 3.X).

The OC method embodies the geometry shown in this chapter. It can be used not only to simply estimate a configuration that maximizes correct classification but also as the foundation for statistically based methods of estimating roll call and legislator parameters. Finally, a nice by-product of its design is that it will almost always find a solution for the perfect voting problem.

Appendix: Proof that if Voting is Perfect in One Dimension, then the First Eigenvector Extracted from the Double-Centered Transformed Agreement Score Matrix has the Same Rank Ordering as the True Data

Notation and Definitions

Let the true ideal points of the p legislators be denoted as $\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2, \ldots, \tilde{\mathbf{X}}_p$. Without loss of generality let the ordering of the true ideal points of the legislators on the dimension from left to right be:

$$\tilde{\mathbf{X}}_1 \leq \tilde{\mathbf{X}}_2 \leq \tilde{\mathbf{X}}_3 \leq \ldots \leq \tilde{\mathbf{X}}_p$$

Let q be the number of *non-unanimous* roll call votes with q > 0 and let the cutpoint for the jth roll call be $\mathbf{Z_j}$. Voting is *perfect*. That is, all legislators are sincere voters and all legislators to the left of a cutpoint vote for the same alternative and all legislators to right of a cutpoint vote for the oppositive alternative. For example, if all legislators to the left of $\mathbf{Z_j}$ vote "Nay", then all legislators to the right of $\mathbf{Z_j}$ vote "Yea". Without loss of generality we can assume that every legislator to the left of $\mathbf{Z_j}$ votes "Yea" and every legislator to the right of $\mathbf{Z_j}$ votes "Nay". That is, the "polarity" of the roll call does not affect the analysis below.

Let k_1 be the number of cutpoints between legislators 1 and 2, k_2 be the number of cutpoints between legislators 2 and 3, and so on, with k_{p-1} being the number of cutpoints between legislators p-1 and p. Hence

$$\mathbf{q} = \sum_{\mathbf{i=1}}^{\mathbf{p-1}} \mathbf{k_i} > \mathbf{0} \qquad \text{(A2.1)}$$

The *agreement score* between two legislators is the simple proportion of roll calls that they vote for the same outcome. Hence, the agreement score between legislators 1 and 2 is simply $\frac{\mathbf{q - k_1}}{\mathbf{q}}$ because 1 and 2 agree on all roll calls except for those with cutpoints between them. Similarly, the agreement score between legislators 1 and 3 is $\frac{\mathbf{q - k_1 - k_2}}{\mathbf{q}}$ and the agreement score between legislators 2 and 3 is $\frac{\mathbf{q - k_2}}{\mathbf{q}}$. In general, for two legislators $\mathbf{X_a}$ and $\mathbf{X_b}$ where a≠b, the agreement score is:

$$\mathbf{A_{ab}} = \frac{\mathbf{q} - \sum_{i=a}^{b-1} \mathbf{k_i}}{\mathbf{q}} \qquad \text{(A2.2)}$$

The agreement scores can be treated as Euclidean distances by simply subtracting them from 1. That is:

$$\mathbf{d_{ab} = 1 - A_{ab} = 1 -} \frac{\mathbf{q} - \sum_{i=a}^{b-1} \mathbf{k_i}}{\mathbf{q}} = \frac{\sum_{i=a}^{b-1} \mathbf{k_i}}{\mathbf{q}} \qquad \text{(A2.3)}$$

These definitions allow me to state the following theorem:

***Theorem*: If Voting is Perfect in One Dimension, then the First Eigenvector Extracted From the Double Centered p by p Matrix of Squared Distances from Equation (A2.3) has at Least the Same Weak Monotone Rank Ordering as the Legislators.**

Proof: The d's computed from equation (A2.3) satisfy the three axioms of distance: they are non-negative because by (A2.2) $\mathbf{0 \le A_{ab} \le 1}$ so that $\mathbf{0 \le d_{ab} \le 1;}$ they are symmetric,

$\mathbf{d_{ab} = d_{ba}}$; and they satisfy the triangle inequality. To see this, consider any triple of points

$\mathbf{X_a < X_b < X_c}$. The distances are:

$$\mathbf{d}_{ab} = \frac{\sum_{i=a}^{b-1} \mathbf{k}_i}{\mathbf{q}} \quad \text{and} \quad \mathbf{d}_{bc} = \frac{\sum_{i=b}^{c-1} \mathbf{k}_i}{\mathbf{q}} \quad \text{and} \quad \mathbf{d}_{ac} = \frac{\sum_{i=a}^{c-1} \mathbf{k}_i}{\mathbf{q}}$$

Hence

$$\mathbf{d_{ac} = d_{ab} + d_{bc}} \qquad \text{(A2.4)}$$

Because all the triangle inequalities are *equalities*, in Euclidean geometry this implies that $\mathbf{X_a}$, $\mathbf{X_b}$, and $\mathbf{X_c}$ *all lie on a straight line* (Borg and Groenen, 1997, ch. 18).

Because all the triangle inequalities are equalities and all triples of points lie on a straight line, the distances computed from (A2.2) can be directly written as distances between points:

$$\mathbf{d}_{ab} = \frac{\sum_{i=a}^{b-1} \mathbf{k}_i}{\mathbf{q}} = |\mathbf{X}_a - \mathbf{X}_b| \qquad \text{(A2.5)}$$

where $\mathbf{d_{aa} = 0}$. The p by p matrix of squared distances is:

$$\mathbf{D} = \begin{bmatrix} 0 & (X_2 - X_1)^2 & . & . & . & (X_p - X_1)^2 \\ (X_1 - X_2)^2 & 0 & . & . & . & (X_p - X_2)^2 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ (X_1 - X_p)^2 & (X_2 - X_p)^2 & . & . & . & 0 \end{bmatrix} \qquad \text{(A2.6)}$$

To recover the **X**'s, simply double-center **D** and perform an eigenvalue-eigenvector decomposition. The first eigenvector is the solution. To see this:

Let the mean of the jth column of **D** be $\mathbf{d}^2_{.j} = \frac{\sum_{i=1}^{p}\mathbf{d}^2_{ij}}{\mathbf{p}} = \mathbf{X}^2_j - 2\mathbf{X}_j\bar{\mathbf{X}} + \frac{\sum_{i=1}^{p}\mathbf{X}^2_i}{\mathbf{p}}$.

Let the mean of the ith row of **D** be $\mathbf{d}^2_{i.} = \frac{\sum_{j=1}^{p}\mathbf{d}^2_{ij}}{\mathbf{p}} = \mathbf{X}^2_i - 2\mathbf{X}_i\bar{\mathbf{X}} + \frac{\sum_{j=1}^{p}\mathbf{X}^2_j}{\mathbf{p}}$.

Let the mean of the matrix **D** be $\mathbf{d}^2_{..} = \frac{\sum_{i=1}^{p}\sum_{j=1}^{p}\mathbf{d}^2_{ij}}{\mathbf{p}^2} = \frac{\sum_{j=1}^{p}\mathbf{X}^2_j}{\mathbf{p}} - 2\bar{\mathbf{X}}^2 + \frac{\sum_{i=1}^{p}\mathbf{X}^2_i}{\mathbf{p}}$.

Where $\bar{\mathbf{X}} = \frac{\sum_{i=1}^{p}\mathbf{X}_i}{\mathbf{p}}$ is the mean of the $\mathbf{X_i}$.

The matrix **D** is double-centered as follows: from each element subtract the row mean, subtract the column mean, add the matrix mean, and divide by –2; that is,

$$\mathbf{y}_{ij} = \frac{(\mathbf{d}^2_{ij} - \mathbf{d}^2_{.j} - \mathbf{d}^2_{i.} + \mathbf{d}^2_{..})}{-2} = (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_j - \bar{\mathbf{X}})$$

This produces the p by p symmetric positive semidefinite matrix **Y**:

$$\mathbf{Y} = \begin{bmatrix} \mathbf{X}_1 - \bar{\mathbf{X}} \\ \mathbf{X}_2 - \bar{\mathbf{X}} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{X}_p - \bar{\mathbf{X}} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 - \bar{\mathbf{X}} & \mathbf{X}_2 - \bar{\mathbf{X}} & . & . & . & \mathbf{X}_p - \bar{\mathbf{X}} \end{bmatrix} \qquad \text{(A2.7)}$$

Because **Y** is symmetric with a rank of one, its eigenvalue-eigenvector decomposition is simply:

$$\mathbf{Y} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{u}_p \end{bmatrix} \lambda_1 \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & . & . & . & \mathbf{u}_p \end{bmatrix} \qquad \text{(A2.8)}$$

Hence, the solution is

$$\begin{bmatrix} \mathbf{X}_1 - \bar{\mathbf{X}} \\ \mathbf{X}_2 - \bar{\mathbf{X}} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{X}_p - \bar{\mathbf{X}} \end{bmatrix} = \sqrt{\lambda_1} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{u}_p \end{bmatrix}$$

Because, without loss of generality, the origin can be placed at zero, that is, $\bar{\mathbf{X}} = \mathbf{0}$, the solution can also be written as:

$$\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{X}_p \end{bmatrix} = \sqrt{\lambda_1} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{u}_p \end{bmatrix} \qquad \textbf{(A2.9)}$$

The points from (A2.9) *exactly reproduce* the distances in (A2.4), the agreement scores in (A2.2), and the original roll call votes. In addition, note that the *mirror image* of the points in (A2.9) (a multiplication by minus one) also exactly reproduces the original roll call votes. Furthermore, for any pair of true legislator ideal points $\tilde{\mathbf{X}}_a$ and $\tilde{\mathbf{X}}_b$ with one or more midpoints between them, $\tilde{\mathbf{X}}_a < \mathbf{Z}_j < \tilde{\mathbf{X}}_b$, the recovered legislator ideal points *must have the same ordering*, $\mathbf{X}_a < \mathbf{X}_b$. If there are no midpoints between $\tilde{\mathbf{X}}_a$ and $\tilde{\mathbf{X}}_b$ **--**

that is, their roll call voting pattern is *identical* -- then the recovered legislator ideal points are identical; $\mathbf{X_a = X_b}$. Hence, if there are cutting points between every pair of adjacent legislators, that is, $\mathbf{k_i} \geq 1$ for i=1, …, p-1, then the rank ordering of the recovered ideal points is the same as the true rank ordering. If some of the $\mathbf{k_i = 0}$, then the recovered ideal points have a weak monotone transformation of the true rank ordering (in other words there are ties, some legislators have the same recovered ideal points).

This completes the proof. **QED**.

Discussion

Note that an interval level set of points is recovered. However, *this is an artifact of the distribution of cutting points*. For example, if $k_1 > k_2$, this has the effect of making $\mathbf{d_{12} > d_{23}}$ even if the true coordinates $\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2, \tilde{\mathbf{X}}_3$ were *evenly spaced*. With perfect one dimensional voting, the legislator configuration is only identified up to a *weak monotone transformation of the true rank ordering*.

The rank ordering can also be recovered from the matrix **Y** given in (A2.7) without performing an eigenvalue-eigenvector decomposition. Note that, with the origin at zero, the diagonal elements of **Y** are simply the legislator coordinates squared. The rank ordering can be recovered by taking the square root of the first diagonal element and then dividing through the first row of the matrix. Note that this sets $\mathbf{X_1 > 0}$ and the remaining points are identified vis a vis $\mathbf{X_1}$.

Chapter Two Notes

[1] Guttman (1944) developed his method to analyze a *cumulative* dimension. For example, a series of questions concerning racial tolerance is asked a set of subjects where the questions are designed to tap ever greater levels of tolerance. Therefore, if a subject answers "Yes" to a question tapping an intermediate level of tolerance then the subject should answer "Yes" to all those questions tapping lower levels of tolerance. Weisberg (1968) discusses in detail the connection between traditional Guttman scaling and the one-dimensional roll call voting problem.

[2] By distance I mean *Euclidean* distance. The following are the most important properties of Euclidean distances. For any pair of points a and b, $d_{aa} = d_{bb} = 0$, $d_{ab} = d_{ba} \geq 0$ – that is, the distance between a point and itself is zero; the distance from a to b is the same as the distance from b to a; and distance must be non-negative. For any triple of points a, b, and c, $d_{ac} \leq d_{ab} + d_{bc}$ (triangle inequality) and $\mathbf{d_{ab}^2 = d_{ac}^2 + d_{bc}^2 - 2d_{ac}d_{bc}\cos\theta}$ where θ is the angle between the vector from c to a and the vector from c to b (cosine law).

[3] Jeff Lewis (20XX) has analyzed interest group ratings of the California state assembly.

[4] Note that if Figure 2.6 is the true configuration but the ADA chooses no cutpoints to its left, then its ratings will produce the correct *weak* rank ordering. That is, legislators 1, 2, and 3 will get the same rating.

[5] Technically, "open" polytope is a contradiction because a polytope has to be bounded by straight lines. I use the expression because it is a convenient way to describe the exterior regions. Note that I could make these open polytopes real polytopes by drawing

straight lines – chords – in place of the arcs in Figure 2.8. In three dimensions the open polytopes can be closed by replacing the corresponding hemispheres with planes.