$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_{11} & \mathbf{x}_{12} & . & . & . & \mathbf{x}_{1s} \\ \mathbf{x}_{21} & \mathbf{x}_{22} & . & . & . & \mathbf{x}_{2s} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \mathbf{x}_{p1} & \mathbf{x}_{p2} & . & . & . & \mathbf{x}_{ps} \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} & . & . & . & \mathbf{z}_{1s} \\ \mathbf{z}_{21} & \mathbf{z}_{22} & . & . & . & \mathbf{z}_{2s} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \mathbf{z}_{q1} & \mathbf{z}_{q2} & . & . & . & \mathbf{z}_{qs} \end{bmatrix}$$

$$\mathbf{J}_p = \begin{bmatrix} 1 \\ 1 \\ 1 \\ : \\ 1 \end{bmatrix} \qquad \mathbf{diag(XX')} = \begin{bmatrix} \sum_{k=1}^{s} x_{1k}^2 \\ \sum_{k=1}^{s} x_{2k}^2 \\ \sum_{k=1}^{s} x_{3k}^2 \\ : \\ \sum_{k=1}^{s} x_{pk}^2 \end{bmatrix} \qquad \mathbf{diag(ZZ')} = \begin{bmatrix} \sum_{k=1}^{s} z_{1k}^2 \\ \sum_{k=1}^{s} z_{2k}^2 \\ \sum_{k=1}^{s} z_{3k}^2 \\ : \\ \sum_{k=1}^{s} z_{qk}^2 \end{bmatrix}$$

The p by q matrix of squared distances between **X** and **Z** (individuals and stimuli) is:

$$
\mathbf{D} = \begin{bmatrix}
\sum_{k=1}^{s}(x_{1k} - z_{1k})^2 & \sum_{k=1}^{s}(x_{1k} - z_{2k})^2 & . & . & . & \sum_{k=1}^{s}(x_{1k} - z_{qk})^2 \\
\sum_{k=1}^{s}(x_{2k} - z_{1k})^2 & \sum_{k=1}^{s}(x_{2k} - z_{2k})^2 & . & . & . & \sum_{k=1}^{s}(x_{2k} - z_{qk})^2 \\
. & . & . & & & . \\
. & . & & . & & . \\
. & . & & & . & . \\
\sum_{k=1}^{s}(x_{pk} - z_{1k})^2 & \sum_{k=1}^{s}(x_{pk} - z_{2k})^2 & . & . & . & \sum_{k=1}^{s}(x_{pk} - z_{qk})^2
\end{bmatrix}
$$

This can be written in matrix algebra as:

$$\mathbf{D = diag(XX')J_q' - 2XZ' + J_p diag(ZZ')' =}$$

$$\mathbf{[diag(XX') \mid -2X \mid J_p]\begin{bmatrix} J_q' \\ \hline Z' \\ \hline diag(ZZ')' \end{bmatrix}}$$

Note that the rank of D, $\boldsymbol{\rho}\mathbf{(D)}$, must be less than or equal to s+2; i.e., $\boldsymbol{\rho}\mathbf{(D) \leq s + 2}$

The equivalent expression for the symmetric matrix of squared distances between the stimuli is:

$$\mathbf{D_z = diag(ZZ')J_q' - 2ZZ' + J_q diag(ZZ')' =}$$

$$\mathbf{[diag(ZZ') \mid -2Z \mid J_q]\begin{bmatrix} J_q' \\ \hline Z' \\ \hline diag(ZZ')' \end{bmatrix}}$$

The Double-Centered Matrix -- D

Let the mean of the jth column of **D** be $\mathbf{d_{.j}^2} = \dfrac{\sum_{i=1}^{p} \mathbf{d_{ij}^2}}{\mathbf{p}} =$

$$\frac{1}{p}\sum_{i=1}^{p}\sum_{k=1}^{s} x_{ik}^2 - \frac{2}{p}\sum_{i=1}^{p}\sum_{k=1}^{s} x_{ik} z_{jk} + \sum_{k=1}^{s} z_{jk}^2$$

Let the mean of the ith row of **D** be $\mathbf{d}_{i.}^{2} = \frac{\sum_{j=1}^{q}\mathbf{d}_{ij}^{2}}{\mathbf{q}} =$

$$\sum_{k=1}^{s}\mathbf{x}_{ik}^{2} - \frac{2}{\mathbf{q}}\sum_{j=1}^{q}\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk} + \frac{1}{\mathbf{q}}\sum_{j=1}^{q}\sum_{k=1}^{s}\mathbf{z}_{jk}^{2}$$

Let the mean of the matrix **D** be $\mathbf{d}_{..}^{2} = \frac{\sum_{i=1}^{p}\sum_{j=1}^{q}\mathbf{d}_{ij}^{2}}{\mathbf{pq}} =$

$$\frac{1}{\mathbf{p}}\sum_{i=1}^{p}\sum_{k=1}^{s}\mathbf{x}_{ik}^{2} - \frac{2}{\mathbf{pq}}\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk} + \frac{1}{\mathbf{q}}\sum_{j=1}^{q}\sum_{k=1}^{s}\mathbf{z}_{jk}^{2}$$

The matrix **D** is **Double-Centered** as follows: from each element subtract the column mean, subtract the row mean, add the matrix mean, and divide by –2; that is,

$$\mathbf{y}_{ij} = \frac{(\mathbf{d}_{ij}^{2} - \mathbf{d}_{.j}^{2} - \mathbf{d}_{i.}^{2} + \mathbf{d}_{..}^{2})}{-2} = \sum_{k=1}^{s}\left[(\mathbf{x}_{ik} - \bar{\mathbf{x}}_{k})(\mathbf{z}_{jk} - \bar{\mathbf{z}}_{k})\right] =$$

$$\frac{1}{-2}\left\{\begin{array}{l}\left(\sum_{k=1}^{s}\mathbf{x}_{ik}^{2} - 2\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk} + \sum_{k=1}^{s}\mathbf{z}_{jk}^{2}\right) - \left(\frac{1}{\mathbf{p}}\sum_{i=1}^{p}\sum_{k=1}^{s}\mathbf{x}_{ik}^{2} - \frac{2}{\mathbf{p}}\sum_{i=1}^{p}\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk} + \sum_{k=1}^{s}\mathbf{z}_{jk}^{2}\right) - \\ \left(\sum_{k=1}^{s}\mathbf{x}_{ik}^{2} - \frac{2}{\mathbf{q}}\sum_{j=1}^{q}\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk} + \frac{1}{\mathbf{q}}\sum_{j=1}^{q}\sum_{k=1}^{s}\mathbf{z}_{jk}^{2}\right) + \\ \left(\frac{1}{\mathbf{p}}\sum_{i=1}^{p}\sum_{k=1}^{s}\mathbf{x}_{ik}^{2} - \frac{2}{\mathbf{pq}}\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk} + \frac{1}{\mathbf{q}}\sum_{j=1}^{q}\sum_{k=1}^{s}\mathbf{z}_{jk}^{2}\right)\end{array}\right\} =$$

$$\left\{\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk} - \frac{1}{\mathbf{p}}\sum_{i=1}^{p}\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk} - \frac{1}{\mathbf{q}}\sum_{j=1}^{q}\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk} + \frac{1}{\mathbf{pq}}\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk}\right\} =$$

$$\left\{\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk} - \sum_{k=1}^{s}\left(\mathbf{z}_{jk}\frac{1}{\mathbf{p}}\sum_{i=1}^{p}\mathbf{x}_{ik}\right) - \sum_{k=1}^{s}\left(\mathbf{x}_{ik}\frac{1}{\mathbf{q}}\sum_{j=1}^{q}\mathbf{z}_{jk}\right) + \sum_{k=1}^{s}\left(\mathbf{z}_{jk}\frac{1}{\mathbf{p}}\sum_{i=1}^{p}\mathbf{x}_{ik}\right)\left(\mathbf{x}_{ik}\frac{1}{\mathbf{q}}\sum_{j=1}^{q}\mathbf{z}_{jk}\right)\right\} =$$

$$\left\{\sum_{k=1}^{s}\mathbf{x}_{ik}\mathbf{z}_{jk} - \sum_{k=1}^{s}\mathbf{z}_{jk}\bar{\mathbf{x}}_{k} - \sum_{k=1}^{s}\mathbf{x}_{ik}\bar{\mathbf{z}}_{k} + \sum_{k=1}^{s}\bar{\mathbf{z}}_{k}\bar{\mathbf{x}}_{k}\right\} =$$

$$\left\{\sum_{k=1}^{s}\left[\mathbf{z}_{jk}\left(\mathbf{x}_{ik} - \bar{\mathbf{x}}\right)\right] - \sum_{k=1}^{s}\left[\bar{\mathbf{z}}\left(\mathbf{x}_{ik} - \bar{\mathbf{x}}\right)\right]\right\} = \sum_{k=1}^{s}\left[(\mathbf{x}_{ik} - \bar{\mathbf{x}}_{k})(\mathbf{z}_{jk} - \bar{\mathbf{z}}_{k})\right]$$

In matrix notation, this produces the p by q matrix **Y** which is equal to the product of a p by s matrix **X*** and a q by s matrix **Z***; namely

$$\mathbf{Y} = \mathbf{X}^*\mathbf{Z}^{*\prime} = \begin{bmatrix} x_{11}-\bar{x}_1 & x_{12}-\bar{x}_2 & . & . & . & x_{1s}-\bar{x}_s \\ x_{21}-\bar{x}_1 & x_{22}-\bar{x}_2 & . & . & . & x_{2s}-\bar{x}_s \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ x_{p1}-\bar{x}_1 & x_{p2}-\bar{x}_2 & . & . & . & x_{ps}-\bar{x}_s \end{bmatrix} \begin{bmatrix} z_{11}-\bar{z}_1 & z_{12}-\bar{z}_2 & . & . & . & z_{1s}-\bar{z}_s \\ z_{21}-\bar{z}_1 & z_{22}-\bar{z}_2 & . & . & . & z_{2s}-\bar{z}_s \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ z_{q1}-\bar{z}_1 & z_{q2}-\bar{z}_2 & . & . & . & z_{qs}-\bar{z}_s \end{bmatrix}^{\prime}$$

Note the reason why this is called a *double* centered matrix. The $\mathbf{x}_i$'s are in a coordinate system *centered at their mean* -- $\bar{\mathbf{x}}$ -- and the $\mathbf{z}_j$'s are in *an entirely different coordinate system* – one *centered at their mean* -- $\bar{\mathbf{z}}$.

The Double-Centered Matrix -- $\mathbf{D_z}$

Let the mean of the jth column of $\mathbf{D_z}$ be $d^2_{.j} = \dfrac{\sum_{m=1}^{q} d^2_{mj}}{q}$

Let the mean of the mth row of $\mathbf{D_z}$ be $d^2_{m.} = \dfrac{\sum_{j=1}^{q} d^2_{mj}}{q}$

Let the mean of the matrix $\mathbf{D_z}$ be $d^2_{..} = \dfrac{\sum_{m=1}^{q}\sum_{j=1}^{q} d^2_{mj}}{q^2}$

The matrix $\mathbf{D_z}$ is **Double-Centered** as follows: from each element subtract the column mean, subtract the row mean, add the matrix mean, and divide by –2; that is,

$$y_{mj} = \frac{(d^2_{mj} - d^2_{.j} - d^2_{m.} + d^2_{..})}{-2} = \sum_{k=1}^{s}(z_{mk} - \bar{z}_k)(z_{jk} - \bar{z}_k)$$

In matrix notation, this produces the q by q matrix **Y** which is equal to the cross-product matrix of the q by s matrix **Z*** times itself; namely

$$\mathbf{Y} = \mathbf{Z}^*\mathbf{Z}^{*\prime} = \begin{bmatrix} z_{11}-\bar{z}_1 & z_{12}-\bar{z}_2 & . & . & . & z_{1s}-\bar{z}_s \\ z_{21}-\bar{z}_1 & z_{22}-\bar{z}_2 & . & . & . & z_{2s}-\bar{z}_s \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ z_{q1}-\bar{z}_1 & z_{q2}-\bar{z}_2 & . & . & . & z_{qs}-\bar{z}_s \end{bmatrix} \begin{bmatrix} z_{11}-\bar{z}_1 & z_{12}-\bar{z}_2 & . & . & . & z_{1s}-\bar{z}_s \\ z_{21}-\bar{z}_1 & z_{22}-\bar{z}_2 & . & . & . & z_{2s}-\bar{z}_s \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ z_{q1}-\bar{z}_1 & z_{q2}-\bar{z}_2 & . & . & . & z_{qs}-\bar{z}_s \end{bmatrix}'$$

Note that this problem is easily solved because, without loss of generality, we can assume that the coordinates have zero means, that is, $\bar{\mathbf{z}} = \mathbf{0}$, and we can use simple eigenvector/eigenvalue decomposition to solve for **Z.**

Example From Old Homework

Below is a matrix of squared distances between 7 points in two dimensions. One of the points is at the origin and the other 6 are arranged symmetrically around it. Double Center the matrix and solve for the coordinates. Show all your computations.

	1	2	3	4	5	6	7
1	0	2	1	2	2	1	2
2	2	0	1	4	8	5	4
3	1	1	0	1	5	4	5
4	2	4	1	0	4	5	8
5	2	8	5	4	0	1	4
6	1	5	4	5	1	0	1
7	2	4	5	8	4	1	0

Find the row, column, and matrix means:

```
Row & Column 1 = (0+2+1+2+2+1+2)/7 = 10/7 = 1.43
Row & Column 2 = (2+0+1+4+8+5+4)/7 = 24/7 = 3.43
Row & Column 3 = (1+1+0+1+5+4+5)/7 = 17/7 = 2.43
Row & Column 4 = (2+4+1+0+4+5+8)/7 = 24/7 = 3.43
Row & Column 5 = (2+8+5+4+0+1+4)/7 = 24/7 = 3.43
Row & Column 6 = (1+5+4+5+1+0+1)/7 = 17/7 = 2.43
Row & Column 7 = (2+4+5+8+4+1+0)/7 = 24/7 = 3.43
```

```
Matrix Mean =
(0+2+1+2+2+1+2+2+0+1+4+8+5+4+1+1+0+1+5+4+5+2+4+1+0+4+5+8+2+
8+5+4+0+1+4+1+5+4+5+1+0+1+2+4+5+8+4+1+0)/49 = 140/49 = 2.86
```

This produces the Double-Centered Matrix:

	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	0	2	1	0	-2	-1	0
3	0	1	1	1	-1	-1	-1
4	0	0	1	2	0	-1	-2
5	0	-2	-1	0	2	1	0
6	0	-1	-1	-1	1	1	1
7	0	0	-1	-2	0	1	2

Answer????

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -1 & -1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 & -2 \\ 0 & -2 & -1 & 0 & 2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -2 & 0 & 1 & 2 \end{bmatrix}$$

So that $\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -1 & -1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$ is an answer.

Remark About Rotations

Any *rotation* of above matrix is also an answer. For example,

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -1 & -1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1.4 & .2 \\ .6 & .8 \\ -.2 & 1.4 \\ -1.4 & -.2 \\ -.6 & -.8 \\ .2 & -1.4 \end{bmatrix}$$

In general: $\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -1 & -1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{\cos\theta} & \mathbf{\sin\theta} \\ \mathbf{-\sin\theta} & \mathbf{\cos\theta} \end{bmatrix}$

Note that: $\begin{bmatrix} \mathbf{\cos\theta} & \mathbf{\sin\theta} \\ \mathbf{-\sin\theta} & \mathbf{\cos\theta} \end{bmatrix} \begin{bmatrix} \mathbf{\cos\theta} & \mathbf{-\sin\theta} \\ \mathbf{\sin\theta} & \mathbf{\cos\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ 0 & \mathbf{1} \end{bmatrix}$

In three dimensions rotation matrices look like this:

$\mathbf{\Gamma_x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathbf{\cos\theta_x} & \mathbf{-\sin\theta_x} \\ 0 & \mathbf{\sin\theta_x} & \mathbf{\cos\theta_x} \end{bmatrix}$, a rotation around the x-axis;

$\mathbf{\Gamma_y} = \begin{bmatrix} \mathbf{\cos\theta_y} & 0 & \mathbf{\sin\theta_y} \\ 0 & 1 & 0 \\ \mathbf{-\sin\theta_y} & 0 & \mathbf{\cos\theta_y} \end{bmatrix}$, a rotation around the y-axis;

$\mathbf{\Gamma_z} = \begin{bmatrix} \mathbf{\cos\theta_z} & \mathbf{-\sin\theta_z} & 0 \\ \mathbf{\sin\theta_z} & \mathbf{\cos\theta_z} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}$, a rotation around the z-axis.

Note that

$\mathbf{\Gamma_x \Gamma_x' = I_3}$

$\mathbf{\Gamma_y \Gamma_y' = I_3}$

$\mathbf{\Gamma_z \Gamma_z' = I_3}$

Where $\mathbf{I_3}$ is a 3 by 3 Identity matrix.

You can get any general rotation by simply multiplying the above matrices; for example:

$\mathbf{\Gamma_x \Gamma_y \Gamma_z}$

$\mathbf{\Gamma_x \Gamma_z}$

$\mathbf{\Gamma_x \Gamma_z \Gamma_y}$

Etc