Solving the Metric Similarities Problem

Recall that our q by s matrix of stimuli coordinates is:

$$\mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} & . & . & . & z_{1s} \\ z_{21} & z_{22} & . & . & . & z_{2s} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ z_{q1} & z_{q2} & . & . & . & z_{qs} \end{bmatrix}$$

And the q by q matrix of squared distances between the q stimuli is:

$$\mathbf{D}_z = \begin{bmatrix} \sum_{k=1}^{s}(z_{1k} - z_{1k})^2 & \sum_{k=1}^{s}(z_{1k} - z_{2k})^2 & . & . & . & \sum_{k=1}^{s}(z_{1k} - z_{qk})^2 \\ \sum_{k=1}^{s}(z_{2k} - z_{1k})^2 & \sum_{k=1}^{s}(z_{2k} - z_{2k})^2 & . & . & . & \sum_{k=1}^{s}(z_{2k} - z_{qk})^2 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \sum_{k=1}^{s}(z_{qk} - z_{1k})^2 & \sum_{k=1}^{s}(z_{qk} - z_{2k})^2 & . & . & . & \sum_{k=1}^{s}(z_{qk} - z_{qk})^2 \end{bmatrix}$$

If there is no error then the solution is:

1. Double-Center $\mathbf{D_z}$

$$\mathbf{Y} = \mathbf{Z}^*\mathbf{Z}^{*\prime} = \begin{bmatrix} z_{11} - \bar{z}_1 & z_{12} - \bar{z}_2 & . & . & . & z_{1s} - \bar{z}_s \\ z_{21} - \bar{z}_1 & z_{22} - \bar{z}_2 & . & . & . & z_{2s} - \bar{z}_s \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ z_{q1} - \bar{z}_1 & z_{q2} - \bar{z}_2 & . & . & . & z_{qs} - \bar{z}_s \end{bmatrix} \begin{bmatrix} z_{11} - \bar{z}_1 & z_{12} - \bar{z}_2 & . & . & . & z_{1s} - \bar{z}_s \\ z_{21} - \bar{z}_1 & z_{22} - \bar{z}_2 & . & . & . & z_{2s} - \bar{z}_s \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ z_{q1} - \bar{z}_1 & z_{q2} - \bar{z}_2 & . & . & . & z_{qs} - \bar{z}_s \end{bmatrix}'$$

2. Compute the eigenvalue-eigenvector decomposition of **Y:**

$$\mathbf{Y} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}'$$

3. Set $\mathbf{Z} = \mathbf{U}\Lambda^{\frac{1}{2}}$

Note that, without loss of generality you can assume that $\bar{\mathbf{z}} = \begin{bmatrix} \bar{\mathbf{z}}_1 \\ \bar{\mathbf{z}}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{\mathbf{z}}_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$

If there is error then the solution is somewhat harder.

1. Define $\mathbf{d}^*_{\mathbf{jm}} = \mathbf{d}_{\mathbf{jm}} + \varepsilon_{\mathbf{jm}} = \sqrt{\sum_{\mathbf{k}=1}^{\mathbf{s}} (\mathbf{z}_{\mathbf{jk}} - \mathbf{z}_{\mathbf{mk}})^2} + \varepsilon_{\mathbf{jm}}$ and set the squared-error loss function to:

$$\mu = \sum_{\mathbf{j}=1}^{\mathbf{q}} \sum_{\mathbf{m}=1}^{\mathbf{q}} \varepsilon^2_{\mathbf{jm}} = \sum_{\mathbf{j}=1}^{\mathbf{q}} \sum_{\mathbf{m}=1}^{\mathbf{q}} \left(\mathbf{d}^*_{\mathbf{jm}} - \mathbf{d}_{\mathbf{jm}}\right)^2 \qquad (1)$$

2. The first derivatives are:

$$\frac{\partial \mu}{\partial \mathbf{z}_{\mathbf{jk}}} = 2\sum_{\mathbf{m}=1}^{\mathbf{q}} \left\{ \left(\mathbf{d}^*_{\mathbf{jm}} - \mathbf{d}_{\mathbf{jm}}\right)\left(-\frac{1}{2}\right)\left[\sum_{\mathbf{k}=1}^{\mathbf{s}} (\mathbf{z}_{\mathbf{jk}} - \mathbf{z}_{\mathbf{mk}})^2\right]^{-\frac{1}{2}} \left(2\left[\mathbf{z}_{\mathbf{jk}} - \mathbf{z}_{\mathbf{mk}}\right]\right) \right\}$$

$$= -2\sum_{\mathbf{m}=1}^{\mathbf{q}} \left\{ \left(\frac{\mathbf{d}^*_{\mathbf{jm}}}{\mathbf{d}_{\mathbf{jm}}} - 1\right)\left(\mathbf{z}_{\mathbf{jk}} - \mathbf{z}_{\mathbf{mk}}\right) \right\} \qquad \mathbf{(2)}$$

3. Setting equal to zero and solving for $\mathbf{z}_{\mathbf{jk}}$**:**

$$\sum_{\mathbf{m}=1}^{\mathbf{q}} \left[\frac{\mathbf{d}^*_{\mathbf{jm}}}{\mathbf{d}_{\mathbf{jm}}} \left(\mathbf{z}_{\mathbf{jk}} - \mathbf{z}_{\mathbf{mk}}\right) \right] - \sum_{\mathbf{m}=1}^{\mathbf{q}} \left(\mathbf{z}_{\mathbf{jk}} - \mathbf{z}_{\mathbf{mk}}\right) = 0$$

Rearranging:

$$-\mathbf{q}\mathbf{z}_{\mathbf{jk}} + \sum_{\mathbf{m}=1}^{\mathbf{q}} \left[\mathbf{z}_{\mathbf{mk}} + \frac{\mathbf{d}^*_{\mathbf{jm}}}{\mathbf{d}_{\mathbf{jm}}} \left(\mathbf{z}_{\mathbf{jk}} - \mathbf{z}_{\mathbf{mk}}\right) \right] = 0$$

Therefore:

$$\hat{\mathbf{z}}_{jk} = \frac{1}{\mathbf{q}}\sum_{m=1}^{q}\left[\mathbf{z}_{mk} + \frac{\mathbf{d}^*_{jm}}{\mathbf{d}_{jm}}\left(\mathbf{z}_{jk} - \mathbf{z}_{mk}\right)\right] \qquad \textbf{(3)}$$

Note that the solution is in the form:

z=f(y,z)

That is, the solution for z is a value such that when it is plugged into **f(y,z)** it produces itself!

Define:

$$\mathbf{z}_{jkm} = \mathbf{z}_{mk} + \frac{\mathbf{d}^*_{jm}}{\mathbf{d}_{jm}}\left(\mathbf{z}_{jk} - \mathbf{z}_{mk}\right) \qquad \textbf{(4)}$$

So that equation (3) can be re-written as:

$$\hat{\mathbf{z}}_{jk} = \frac{1}{\mathbf{q}}\sum_{m=1}^{q}\mathbf{z}_{jkm} \qquad \textbf{(5)}$$

Using equation (4), note that the *point* $\mathbf{z}_{j,m}$ is:

$$\mathbf{z}_{j,m} = \begin{bmatrix} \mathbf{z}_{m1} + \frac{\mathbf{d}^*_{jm}}{\mathbf{d}_{jm}}\left(\mathbf{z}_{j1} - \mathbf{z}_{m1}\right) \\ \mathbf{z}_{m2} + \frac{\mathbf{d}^*_{jm}}{\mathbf{d}_{jm}}\left(\mathbf{z}_{j2} - \mathbf{z}_{m2}\right) \\ . \\ . \\ . \\ \mathbf{z}_{ms} + \frac{\mathbf{d}^*_{jm}}{\mathbf{d}_{jm}}\left(\mathbf{z}_{js} - \mathbf{z}_{ms}\right) \end{bmatrix} = \mathbf{z}_{m} + \frac{\mathbf{d}^*_{jm}}{\mathbf{d}_{jm}}\left(\mathbf{z}_{j} - \mathbf{z}_{m}\right) \qquad (6)$$

Where $\mathbf{z_j} = \begin{bmatrix} \mathbf{z_{j1}} \\ \mathbf{z_{j2}} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{z_{js}} \end{bmatrix}$ and $\mathbf{z_m} = \begin{bmatrix} \mathbf{z_{m1}} \\ \mathbf{z_{m2}} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{z_{ms}} \end{bmatrix}$ are points and $\frac{\mathbf{d^*_{jm}}}{\mathbf{d_{jm}}}$ is a scalar. ***Equation (6) is the basic equation of a line that passes through*** $\mathbf{z_j}$ ***and*** $\mathbf{z_m}$**!** The general formula for a line equation is:

$$\mathbf{Y(t)=A+t(B-A)} \qquad \mathbf{(7)}$$

Where **A and B** are points and **t** is a scalar. Note that if **0<t<1** then equation (7) defines a line that runs between points **A** and **B.**

Once specific values are plugged into equation (6) then the solution for the point

$\hat{\mathbf{z}}_{\mathbf{j}} = \begin{bmatrix} \hat{\mathbf{z}}_{\mathbf{j1}} \\ \hat{\mathbf{z}}_{\mathbf{j2}} \\ \cdot \\ \cdot \\ \cdot \\ \hat{\mathbf{z}}_{\mathbf{js}} \end{bmatrix}$ is simply ***the centroid of the q*** $\mathbf{z_{j,m}}$ ***points*!**

Finally, note that the squared distance between the points $\mathbf{z_j}$ and $\mathbf{z_{j,m}}$ is:

$$\sum_{\mathbf{k}=1}^{\mathbf{s}}\left(\mathbf{z_{jk}}-\mathbf{z_{jkm}}\right)^2 = \sum_{\mathbf{k}=1}^{\mathbf{s}}\left[\mathbf{z_{jk}}-\left(\mathbf{z_{mk}}+\frac{\mathbf{d^*_{jm}}}{\mathbf{d_{jm}}}\left(\mathbf{z_{jk}}-\mathbf{z_{mk}}\right)\right)\right]^2 =$$

$$\sum_{\mathbf{k}=1}^{\mathbf{s}}\left[\left(\mathbf{z_{jk}}-\mathbf{z_{mk}}\right)\left(1-\frac{\mathbf{d^*_{jm}}}{\mathbf{d_{jm}}}\right)\right]^2 = \frac{\left(\mathbf{d_{jm}}-\mathbf{d^*_{jm}}\right)^2}{\mathbf{d^2_{jm}}}\left(\sum_{\mathbf{k}=1}^{\mathbf{s}}\left(\mathbf{z_{jk}}-\mathbf{z_{mk}}\right)^2\right) = \varepsilon^2_{\mathbf{jm}} \qquad (8)$$

So that the squared error is represented directly on the s-dimensional hyperplane (see below).

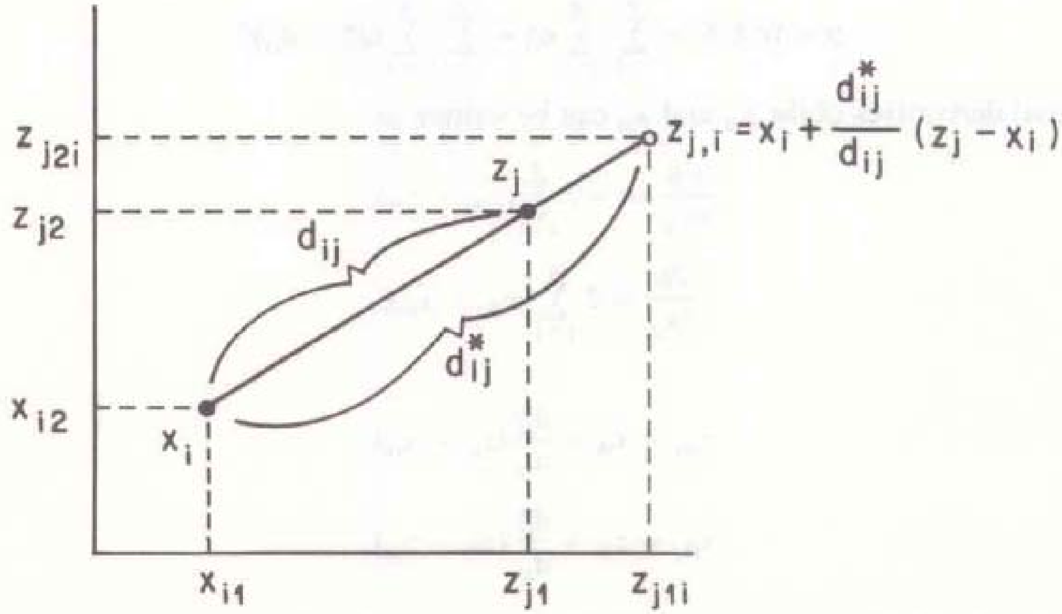

FIGURE 1
Parametric equation of a straight line.