Derivation of the Logit Probability

Utility function for Yea and Nay choices:

$$\mathbf{U_{iy} = e^{-d_{iy}^2} + \varepsilon_{iy} \quad and \quad U_{in} = e^{-d_{in}^2} + \varepsilon_{in}} \qquad \mathbf{(1)}$$

where $\mathbf{d_{iy}^2}$ and $\mathbf{d_{in}^2}$ are the squared distances from the ith legislator to the Yea and Nay choices and the ε are distributed as the logarithm of the inverse of an exponential variable (Dhrymes, 1978, p. 342). Namely

$$\mathbf{f(\varepsilon) = e^{-\varepsilon}e^{-e^{-\varepsilon}}, \; -\infty < \varepsilon < +\infty} \qquad \mathbf{(2)}$$

The probability that the legislator will choose the Yea alternative is:

$$\mathbf{P(U_{iy} > U_{in}) = P(e^{-d_{iy}^2} + \varepsilon_{iy} > e^{-d_{in}^2} + \varepsilon_{in}) = P(e^{-d_{iy}^2} - e^{-d_{in}^2} > \varepsilon_{in} - \varepsilon_{iy}) =}$$

$$\mathbf{P(\varepsilon_{in} - \varepsilon_{iy} < e^{-d_{iy}^2} - e^{-d_{in}^2}) = P(\varepsilon_{iy} - \varepsilon_{in} > e^{-d_{in}^2} - e^{-d_{iy}^2})} \qquad \mathbf{(3)}$$

In order to get the distribution of $\mathbf{\varepsilon_{iy} - \varepsilon_{in}}$ set up the joint density and then do a change of variables (note that the distribution of $\mathbf{\varepsilon_{in} - \varepsilon_{iy}}$ will be the same as the distribution of $\mathbf{\varepsilon_{iy} - \varepsilon_{in}}$):

$$\mathbf{f(\varepsilon_{iy}, \varepsilon_{in}) = e^{-(\varepsilon_{iy} + \varepsilon_{in})}e^{-(e^{-\varepsilon_{iy}} + e^{-\varepsilon_{in}})}} \qquad \mathbf{(4)}$$

Set $y = \varepsilon_{iy} - \varepsilon_{in}$ **and** $z = \varepsilon_{in}$

Hence $\varepsilon_{iy} = y + z$ **and** $\varepsilon_{in} = z$

and the Jacobian is:

$$J = \begin{vmatrix} \frac{\partial \varepsilon_{iy}}{\partial y} & \frac{\partial \varepsilon_{iy}}{\partial z} \\ \frac{\partial \varepsilon_{in}}{\partial y} & \frac{\partial \varepsilon_{in}}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

Hence

$$f(y+z\ ,\ z) = e^{-(y+2z)} e^{-(e^{-(y+z)} + e^{-z})} = e^{-y} e^{-2z} e^{-[e^{-z}(1+e^{-y})]}$$

To get the distribution of $y = \varepsilon_{iy} - \varepsilon_{in}$ integrate out **z**:

$$\int_{-\infty}^{+\infty} e^{-y} e^{-2z} e^{-[e^{-z}(1+e^{-y})]} dz \qquad \textbf{(5)}$$

This requires another change in variables:

Set $v = e^{-z}(1 + e^{-y})$

Note that $0 < v < \infty$ **because** $0 < e^{-z} < \infty$ **as** $-\infty < z < \infty$

Hence, $\ln(v) = -z + \ln(1+e^{-y})$

and $z = \ln(1+e^{-y}) - \ln(v)$

Therefore, $\frac{\partial z}{\partial v} = -\frac{1}{v}$ **and** $J = \left|\frac{\partial z}{\partial v}\right| = \frac{1}{v}$

Hence

$$\int_{-\infty}^{+\infty} e^{-y} e^{-2z} e^{-[e^{-z}(1+e^{-y})]} dz = e^{-y} \int_{0}^{+\infty} e^{-2\ln(1+e^{-y})} e^{2\ln(v)} e^{-\{e^{-[\ln(1+e^{-y})-\ln(v)]}[1+e^{-y}]\}} \frac{1}{v} dv =$$

$$e^{-y} \int_{0}^{+\infty} (1+e^{-y})^{-2} v^{2} e^{-\{(1+e^{-y})^{-1} v[1+e^{-y}]\}} \frac{1}{v} dv = e^{-y} (1+e^{-y})^{-2} \int_{0}^{+\infty} v^{2} e^{-v} \frac{1}{v} dv =$$

$$e^{-y} (1+e^{-y})^{-2} \int_{0}^{+\infty} v e^{-v} dv = e^{-y} (1+e^{-y})^{-2} \Gamma(2) = e^{-y} (1+e^{-y})^{-2} \qquad (6)$$

Therefore, $$f(\varepsilon_{iy} - \varepsilon_{in}) = f(y) = \frac{e^{-y}}{(1+e^{-y})^{2}} , \qquad (7)$$

To get the distribution function:

$$F(y<t) = \int_{-\infty}^{t} \frac{e^{-y}}{(1+e^{-y})^{2}} dy = \left. \frac{1}{1+e^{-y}} \right|_{-\infty}^{t} = \frac{1}{1+e^{-t}}$$

Hence

$$P(\varepsilon_{in} - \varepsilon_{iy} < e^{-d_{iy}^{2}} - e^{-d_{in}^{2}}) = \frac{1}{1 + e^{-(e^{-d_{iy}^{2}} - e^{-d_{in}^{2}})}} \qquad (8)$$