Eigenvalues and Eigenvectors (Characteristic values and Characteristic vectors) of a symmetric matrix.

Let **A** be an n by n symmetric matrix, let **u** be an n by 1 length vector (**u** ≠ **0**), and let λ be a constant (a ***scalar***). Then:

a. $\mathbf{A\underline{u}} = \lambda\mathbf{\underline{u}}$ or $(\mathbf{A} - \lambda\mathbf{I})\mathbf{\underline{u}} = \mathbf{\underline{0}}$.

b. Given that **u** ≠ **0** it must be the case that:

c. $|\mathbf{A} - \lambda\mathbf{I}| = \mathbf{0}$**,** that is, the determinant of the matrix must be zero. The ***characteristic equation*** is the polynomial in λ. For a two by two matrix **A** this is:

d. $$|\mathbf{A} - \lambda\mathbf{I}| = \left|\begin{bmatrix} \mathbf{a}_{11} - \lambda & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} - \lambda \end{bmatrix}\right| = (\mathbf{a}_{11} - \lambda)(\mathbf{a}_{22} - \lambda) - \mathbf{a}_{12}\mathbf{a}_{22} = \mathbf{0}\,.$$ Now,

since **A** is a symmetric matrix ($\mathbf{a}_{12} = \mathbf{a}_{21}$), we can write the characteristic equation as:

e. $\lambda^2 - \lambda(\mathbf{a}_{11} + \mathbf{a}_{22}) + \mathbf{a}_{11}\mathbf{a}_{22} - \mathbf{a}_{12}^2 = \mathbf{0}$. This is a standard polynomial of degree two and therefore has two roots -- λ_1 **and** λ_2 – these are the ***eigenvalues* of A**. Corresponding to these two eigenvalues are two ***eigenvectors*** – $\mathbf{\underline{u}_1}$ and $\mathbf{\underline{u}_2}$. We can solve for these two eigenvectors by substituting first λ_1 and then λ_2 into the equation above; namely:

f. $$(\mathbf{A} - \lambda_1\mathbf{I}) = \begin{bmatrix} \mathbf{a}_{11} - \lambda_1 & \mathbf{a}_{12} \\ \mathbf{a}_{12} & \mathbf{a}_{22} - \lambda_1 \end{bmatrix}\begin{bmatrix} \mathbf{u}_{11} \\ \mathbf{u}_{21} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$ **and**

g. $$(\mathbf{A} - \lambda_2\mathbf{I}) = \begin{bmatrix} \mathbf{a}_{11} - \lambda_2 & \mathbf{a}_{12} \\ \mathbf{a}_{12} & \mathbf{a}_{22} - \lambda_2 \end{bmatrix}\begin{bmatrix} \mathbf{u}_{12} \\ \mathbf{u}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

h. Let $\mathbf{U} = \begin{bmatrix} \mathbf{u}_{11} & \mathbf{u}_{12} \\ \mathbf{u}_{21} & \mathbf{u}_{22} \end{bmatrix}$**.,** that is, the ***columns*** of **U** are the eigenvectors. **U *is an orthogonal matrix***. That is, **U'U = UU' = I.**

i. Let $\Lambda = \begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & \lambda_2 \end{bmatrix}$ be a ***diagonal*** matrix with the eigenvalues on the diagonal. Therefore, the symmetric matrix, **A**, can be written as:

j. **A = UΛU'**. Also, note that: **U'AU = Λ.**

Within R the command is:

ev <- eigen(TT)

where **ev** is a data structure containing the eigenvalues and eigenvectors **–**

ev$value and **ev$vector**

For the Double-Centered matrix example the eigenvectors are:

$$\mathbf{U}=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.408 & 0.5 & 0.702 & 0 & 0 & 0 & 0.302 \\ 0.408 & 0 & -0.108 & 0.025 & 0 & -0.855 & -0.300 \\ 0.408 & -0.5 & 0.025 & 0.719 & 0 & 0.136 & 0.218 \\ -0.408 & -0.5 & 0.697 & -0.029 & 0 & -0.200 & -0.239 \\ -0.408 & 0 & -0.099 & 0.084 & 0 & -0.454 & 0.781 \\ -0.408 & 0.5 & 0.020 & 0.689 & 0 & -0.064 & -0.322 \end{bmatrix}$$

$$\mathbf{\Lambda}=\begin{bmatrix} 6 & & & & & & \\ & 4 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & -2 & -1 & 0 \\
0 & 1 & 1 & 1 & -1 & -1 & -1 \\
0 & 0 & 1 & 2 & 0 & -1 & -2 \\
0 & -2 & -1 & 0 & 2 & 1 & 0 \\
0 & -1 & -1 & -1 & 1 & 1 & 1 \\
0 & 0 & -1 & -2 & 0 & 1 & 2
\end{bmatrix} =
$$

$$
\begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0.408} & \mathbf{0.5} & \mathbf{0.702} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0.302} \\
\mathbf{0.408} & \mathbf{0} & \mathbf{-0.108} & \mathbf{0.025} & \mathbf{0} & \mathbf{-0.855} & \mathbf{-0.300} \\
\mathbf{0.408} & \mathbf{-0.5} & \mathbf{0.025} & \mathbf{0.719} & \mathbf{0} & \mathbf{0.136} & \mathbf{0.218} \\
\mathbf{-0.408} & \mathbf{-0.5} & \mathbf{0.697} & \mathbf{-0.029} & \mathbf{0} & \mathbf{-0.200} & \mathbf{-0.239} \\
\mathbf{-0.408} & \mathbf{0} & \mathbf{-0.099} & \mathbf{0.084} & \mathbf{0} & \mathbf{-0.454} & \mathbf{0.781} \\
\mathbf{-0.408} & \mathbf{0.5} & \mathbf{0.020} & \mathbf{0.689} & \mathbf{0} & \mathbf{-0.064} & \mathbf{-0.322}
\end{bmatrix}
\begin{bmatrix}
6 & & & & & & \\
 & 4 & & & & & \\
 & & 0 & & & & \\
 & & & 0 & & & \\
 & & & & 0 & & \\
 & & & & & 0 & \\
 & & & & & & 0
\end{bmatrix} \mathrm{X}
$$

$$
\begin{bmatrix}
\mathbf{0} & \mathbf{0.408} & \mathbf{0.408} & \mathbf{0.408} & \mathbf{-0.408} & \mathbf{-0.408} & \mathbf{-0.408} \\
\mathbf{0} & \mathbf{0.5} & \mathbf{0} & \mathbf{-0.5} & \mathbf{-0.5} & \mathbf{0} & \mathbf{0.5} \\
\mathbf{0} & \mathbf{0.702} & \mathbf{-0.108} & \mathbf{0.025} & \mathbf{0.697} & \mathbf{-0.099} & \mathbf{0.020} \\
\mathbf{0} & \mathbf{0} & \mathbf{0.025} & \mathbf{0.719} & \mathbf{-0.029} & \mathbf{0.084} & \mathbf{0.689} \\
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{-0.855} & \mathbf{0.136} & \mathbf{-0.200} & \mathbf{-0.454} & \mathbf{-0.064} \\
\mathbf{0} & \mathbf{0.302} & \mathbf{-0.300} & \mathbf{0.218} & \mathbf{-0.239} & \mathbf{0.781} & \mathbf{-0.322}
\end{bmatrix}
$$